Magic Square of Squares

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1 Introduction

As written by Terry Moriarty published on http://magicsqr.byethost8.com/introMSoS.htm we know that a 3×3 magic square of squares can be represented as

$j^2\left(p^2-2pq-q^2\right)^2$	$k^2\left(m^2+2mn-n^2\right)^2$	$l^2\left(r^2+s^2\right)^2$
$l^2\left(r^2+2rs-s^2\right)^2$	$j^2\left(p^2+q^2\right)^2$	$k^2\left(m^2-2mn-n^2\right)^2$
$k^2\left(m^2+n^2\right)^2$	$l^2\left(r^2-2rs-s^2\right)^2$	$j^2\left(p^2+2pq-q^2\right)^2$

where $m > n > 0$, $p > q > 0$, $r > s > 0$, $\{j, k, l\} > 0$.

We deduce the following two simple constraints.

Constraint (i) The following three formulaes need to be equal

$$4k^2mn\left(m^2-n^2\right) \tag{1}$$
$$4j^2pq\left(p^2-q^2\right) \tag{2}$$
$$4l^2rs\left(r^2-s^2\right) \tag{3}$$

Constraint (ii) The following two formulaes need to be equal

$$j^2\left(p^2-2pq-q^2\right)^2-k^2\left(m^2+2mn-n^2\right)^2 \tag{4}$$
$$l^2\left(r^2-2rs-s^2\right)^2-j^2\left(p^2+2pq-q^2\right)^2 \tag{5}$$

Terry Moriarty claims that any solution simultaneously satisfying these two constraints would give us a magic square of squares.

Let us define

$$a = p^2-2pq-q^2 \tag{6}$$
$$b = m^2+2mn-n^2 \tag{7}$$
$$c = r^2-2rs-s^2 \tag{8}$$
$$d = p^2+2pq-q^2 \tag{9}$$

Then constraint (ii) can be rewritten as

$$j^2 \cdot a^2 - k^2 \cdot b^2 = l^2 \cdot c^2 - j^2 \cdot d^2 \tag{10}$$

2 A possible solution

Using a small program, I have found the following possible solution.

Let there be

$$\begin{aligned} k = 1,\ m = 15,\ n = 3 \\ j = 2,\ p = 9,\ q = 6 \\ l = 9,\ r = 5,\ s = 1 \end{aligned}$$

Then formulae (1) becomes

$$\begin{aligned} 4k^2mn\left(m^2 - n^2\right) &= 4 \cdot 1^2 \cdot 15 \cdot 3 \cdot \left(15^2 - 3^2\right) \\ &= 180 \cdot (225 - 9) \\ &= 38\,880 \end{aligned}$$

Formulae (2) becomes

$$\begin{aligned} 4j^2pq\left(p^2 - q^2\right) &= 4 \cdot 2^2 \cdot 9 \cdot 6 \cdot \left(9^2 - 6^2\right) \\ &= 864 \cdot (81 - 36) \\ &= 38\,880 \end{aligned}$$

And formulae (3) becomes

$$\begin{aligned} 4l^2rs\left(r^2 - s^2\right) &= 4 \cdot 9^2 \cdot 5 \cdot 1 \cdot \left(5^2 - 1^2\right) \\ &= 1\,620 \cdot (25 - 1) \\ &= 38\,880 \end{aligned}$$

Therefore constraint (i) is satisfied.

Now we can verify the constraint (ii). To do this, let us first calculate a, b, c and d as defined in equations (6)-(9).

$$\begin{aligned} a &= p^2 - 2pq - q^2 = 9^2 - 2 \cdot 9 \cdot 6 - 6^2 = -63 \\ b &= m^2 + 2mn - n^2 = 15^2 + 2 \cdot 15 \cdot 3 - 3^2 = 306 \\ c &= r^2 - 2rs - s^2 = 5^2 - 2 \cdot 5 \cdot 1 - 1^2 = 14 \\ d &= p^2 + 2pq - q^2 = 9^2 + 2 \cdot 9 \cdot 6 - 6^2 = 153 \end{aligned}$$

Formulae (4) becomes

$$\begin{aligned} j^2 \cdot a^2 - k^2 \cdot b^2 &= 2^2 \cdot (-63)^2 - 1^2 \cdot 306^2 \\ &= -77\,760 \end{aligned}$$

Formulae (5) becomes

$$\begin{aligned} l^2 \cdot c^2 - j^2 \cdot d^2 &= 9^2 \cdot 14^2 - 2^2 \cdot 153^2 \\ &= -77\,760 \end{aligned}$$

So constraint (ii) is satisfied too. Heureka - we have a magic square of squares.

3 The problem

The magic square of squares with the given solution is as follows.

126^2	306^2	234^2
306^2	234^2	126^2
234^2	126^2	306^2

It is indeed a magic square of squares, but unfortunately the numbers are not unique. Hence I guess we need to adapt the constraints.