Supplement to the article
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Summary

A) Lucas's 3×3 semi-magic squares of squares

$(p^2 + q^2 - r^2 - s^2)^2$	$[2(qr + ps)]^2$	$[2(qs - pr)]^2$
$[2(qr - ps)]^2$	$(p^2 - q^2 + r^2 - s^2)^2$	$[2(rs + pq)]^2$
$[2(qs + pr)]^2$	$[2(rs - pq)]^2$	$(p^2 - q^2 - r^2 + s^2)^2$

EL1 from The Mathematical Intelligencer article

The 3 rows and 3 columns have the same magic sum:

- $S2 = (p^2 + q^2 + r^2 + s^2)^2$.

B) Euler's 4×4 magic squares of squares

$(+ap+bq+cr+ds)^2$	$(+ar-bs-cp+dq)^2$	$(-as-br+cq+dp)^2$	$(+aq-bp+cs-dr)^2$
$(-aq+bp+cs-dr)^2$	$(+as+br+cq+dp)^2$	$(+ar-bs+cp-dq)^2$	$(+ap+bq-cr-ds)^2$
$(+ar+bs-cp-dq)^2$	$(-ap+bq-cr+ds)^2$	$(+aq+bp+cs+dr)^2$	$(+as-br-cq+dp)^2$
$(-as+br-cq+dp)^2$	$(-aq-bp+cs+dr)^2$	$(-ap+bq+cr-ds)^2$	$(+ar+bs+cp+dq)^2$

LE3 from The Mathematical Intelligencer article

The 4 rows and 4 columns have the same magic sum:

- $S2 = (a^2 + b^2 + c^2 + d^2)(p^2 + q^2 + r^2 + s^2)$.

Two supplemental conditions are given to get the two magic diagonals:

- $pr + qs = 0$,
- $a / c = [- d(pq + rs) - b(ps + qr)] / [b(pq + rs) + d(ps + qr)]$.

C) Using prime numbers

A) Lucas's 3×3 semi-magic squares of squares

A1) The full list of examples of Lucas's family, producing distinct numbers, six magic lines, and a magic sum $\leq 100^2$:

(p, q, r, s)	Magic sum
(1, 2, 4, 6)*	57^2 (the *LE1cb* square)
(1, 2, 3, 7)*	63^2
(2, 3, 4, 6)	65^2 (the *AB2* square)
(1, 3, 5, 6)*	71^2
(1, 2, 5, 7)	79^2
(2, 4, 5, 6)**	81^2 (the *EL2* square)
(1, 2, 4, 8)	85^2
(1, 4, 5, 7)	91^2
(2, 3, 4, 8)	93^2
(1, 3, 6, 7)	95^2
(1, 3, 5, 8)*	99^2
(3, 4, 5, 7)	99^2 (a different square with the same sum)

and of course all the possible permutations of positions and signs of p, q, r, s.

* : The four examples published by Euler in 1770. They were presented slightly differently, Euler using rational numbers: the nine numbers were signed, not squared, and they were divided by $p^2+q^2+r^2+s^2$.

** : The example published by Lucas in 1876.

A2) The full list of examples of Lucas's family, producing distinct numbers, seven magic lines, and a magic sum $\leq 2000^2$ are:

(p, q, r, s)	Magic sum
(1, 3, 4, 11)	147^2 (the *MS1* square)
(3, 5, 8, 14) *(A)*	294^2 (the *MS2* square)
(4, 9, 11, 17)	507^2
(2, 6, 8, 22)	588^2 (three identical permuted squares)
(3, 11, 13, 17)	588^2 (three identical permuted squares)
(5, 9, 11, 19)	588^2 (three identical permuted squares)
(7, 8, 15, 26) *(B)*	1014^2
(8, 11, 13, 27)	1083^2
(6, 10, 16, 28) *(C)*	1176^2
(3, 9, 12, 33)	1323^2

and of course all the possible permutations of positions and signs of p, q, r, s.

None of these examples were published by Lucas or Euler.

Some other (p, q, r, s) produce only 6 magic lines, but their cells can easily be permuted to get some of the above squares with 7 magic lines:

Squares generated by (1, 7, 10, 12) and (2, 4, 7, 15) can be permuted to get the square generated by *(A)*

Squares generated by (2, 13, 20, 21) and (5, 6, 13, 28) can be permuted to get the square generated by *(B)*

Squares generated by (2, 14, 20, 24) and (4, 8, 14, 30) can be permuted to get the square generated by *(C)*

B) Euler's 4×4 magic squares of squares

The full list of examples of Euler's family, producing distinct numbers, ten magic lines (by definition of Euler's family), and a magic sum ≤ 10000 are:

(*a*, *b*, *c*, *d*, *p*, *q*, *r*, *s*)	Magic sum
(2, 3, 5, 0, 1, 2, 8, –4) *(D)*	3230 (the *CB1* magic square)
(1, 2, 3, 4, 2, 5, 10, –4)	4350
(1, 4, 6, 1, 1, 2, 8, –4)	4590
(2, 5, 5, 0, 1, 2, 8, –4) *(D)*	4590 (a different square with the same sum)
(5, 2, 9, 0, 2, 3, 6, –4) *(E)*	7150 (the Benneton square)
(5, 3, 9, 0, 2, 3, 6, –4) *(E)*	7475
(2, 8, 5, 0, 1, 2, 8, –4) *(D)*	7905
(5, 5, 9, 0, 2, 3, 6, –4) *(E)**	8515 (a permutation of the *LE2* magic square)
(5, 6, 9, 0, 2, 3, 6, –4) *(E)*	9230
(4, 1, 10, 0, 1, 2, 8, –4)	9945

and of course all the possible permutations of positions and signs of *a*, *b*, *c*, *d*, *p*, *q*, *r*, *s*, like the only example * given by Euler, the magic square *LE2* sent to Lagrange.

(2, *k*, 5, 0, 1, 2, 8, –4) *(D)* and (5, *k*, 9, 0, 2, 3, 6, –4) *(E)* give these two very nice sub-families (*CB2*) and (*CB15*) of magic squares of squares. The only limitation: the 16 generated numbers are not always distinct for every *k*.

$(2k + 42)^2$	$(4k + 11)^2$	$(8k - 18)^2$	$(k + 16)^2$
$(k - 24)^2$	$(8k + 2)^2$	$(4k + 21)^2$	$(2k - 38)^2$
$(4k - 11)^2$	$(2k - 42)^2$	$(k - 16)^2$	$(8k + 18)^2$
$(8k - 2)^2$	$(k + 24)^2$	$(2k + 38)^2$	$(4k - 21)^2$

CB2 from The Mathematical Intelligencer article.
A sub-family of Euler's magic square of squares, $S2 = 85(k^2 + 29)$.
Its 16 numbers are distinct for $k = 3, 5, 8, \ldots$

$(3k + 64)^2$	$(4k + 12)^2$	$(6k - 47)^2$	$(2k + 21)^2$
$(2k - 51)^2$	$(6k + 7)^2$	$(4k + 48)^2$	$(3k - 44)^2$
$(4k - 12)^2$	$(3k - 64)^2$	$(2k - 21)^2$	$(6k + 47)^2$
$(6k - 7)^2$	$(2k + 51)^2$	$(3k + 44)^2$	$(4k - 48)^2$

CB15. Another sub-family of Euler's magic square of squares, $S2 = 65(k^2 + 106)$.
Its 16 numbers are distinct for $k = 2, 3, 5, \ldots$

There are only 6 other magic squares of squares, with a magic sum ≤10000, that are not part of the Euler's family:

Magic sum
2823 (the *AB3* magic square)
4875
6462
7150 (a square different from Benneton's square)
7735
9775

C) Using prime numbers

What about the magic squares of squares problem, if only squares of distinct prime numbers are allowed? The research is more difficult, but it gives the following (*CB16*) and (*CB17*) 4×4 results.

29	293	641	227
277	659	73	181
643	101	337	109
241	137	139	673

CB16. The smallest 4×4 semi-bimagic square of prime numbers. $S1 = 1190$, $S2 = 549100$.

29^2	191^2	673^2	137^2
71^2	647^2	139^2	257^2
277^2	211^2	163^2	601^2
653^2	97^2	101^2	251^2

CB17. The smallest 4×4 magic square of squares of prime numbers, $S2 = 509020$.

There are some interesting similarities if you analyze and compare the two squares:

- The smallest prime number is the same in the two squares, 29, and it is located in the same place, the first cell.
- The biggest prime number is also the same in the two squares, 673.
- The same pair of twin prime numbers is used, 137 and 139.
- Two other identical numbers are used in the two squares, 101 and 277.
- There is a big gap in the numbers used in the two squares:
  - 304 in the (*CB16*) square → nothing is used between 337 and 641
  - 324 in the (*CB17*) square → nothing is used between 277 and 601.

See prime puzzles 287 and 288 on Carlos Rivera's Web site at http://www.primepuzzles.net

11^2	23^2	53^2	139^2	107^2
13^2	103^2	149^2	31^2	17^2
71^2	137^2	47^2	67^2	61^2
113^2	59^2	41^2	97^2	83^2
127^2	29^2	73^2	7^2	109^2

CB18. The smallest 5×5 magic square of squares of prime numbers, $S2 = 34229$.

Two open problems from the ten published in *The Mathematical Intelligencer* article:

- Open problem 4. Construct a bimagic square of prime numbers
- Open problem 6. Construct a magic square of cubes of prime numbers