Smallest magic squares of triangular numbers.
Expanded solution to the 1941 Problem E 496
in *The American Mathematical Monthly*

by Christian Boyer, www.multimagie.com/indexengl.htm

(expanded version of the solution published in
The American Mathematical Monthly, Vol 114, N 8, October 2007, p 745-746)
V2.0, June 2008, with additions (magic squares of pentagonal numbers, by Lee Morgenstern)

An old problem proposed in 1941, solved in 2007

In 1941, **Royal Vale Heath** proposed this short problem in *The American Mathematical Monthly* [4] when **H. S. M. Coxeter** was in charge of the "Problems and Solutions" column:

> "What is the smallest value of n for which the n^2 triangular numbers 0, 1, 3, 6, 10, …, $n^2(n^2-1)/2$ can be arranged to form a magic square?"

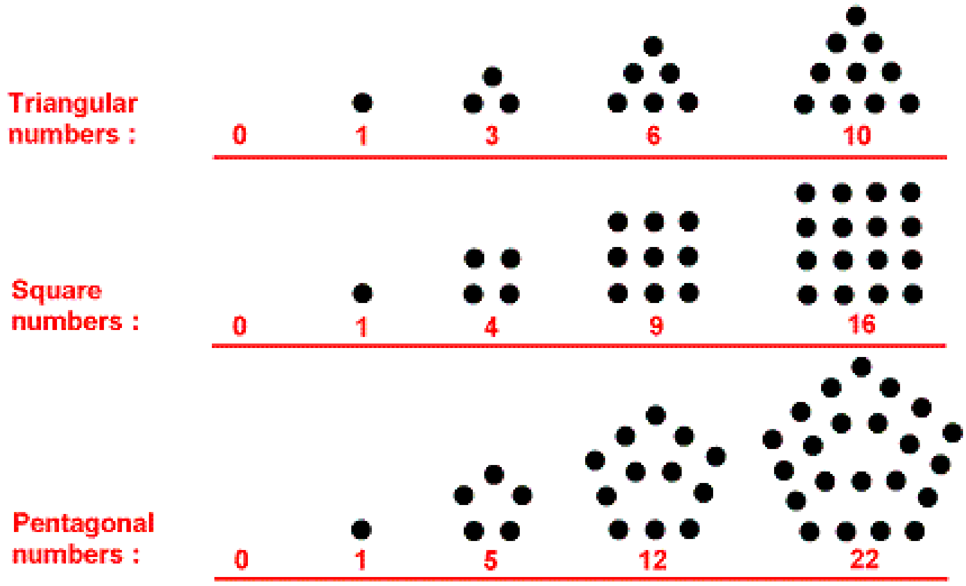
This problem remained unsolved. Here is the solution we found in April 2007… 66 years later:

$n = 6$.

The samples were not easy to find, but are easy for the reader to check, as a factorization problem.

Triangular and polygonal numbers

This figure will help us understand what triangular numbers are, and more generally polygonal numbers.



First polygonal numbers

Relationship to bimagic squares

In his partial solution [5] published in the *Monthly* in 1942, **R. V. Heath** remarked that a bimagic square (magic square which is still magic after the original entries are all squared [2a]) can be directly used to construct a magic square of triangular numbers:

> "Clearly, the magic property will still be retained if each of the original numbers is subtracted from its square. The resulting numbers are all even, and their halves are the triangular numbers"

Using a bimagic square of order 8 found in the well-known book [1, p. 212] by **W. W. Rouse Ball** and initially constructed by **H. Schots** in 1931 [8, p. 357], Heath [5] built a magic square of 64 triangular numbers and with it showed that $n \leq 8$, but the smallest possible n remained unknown, as he said:

> "But it remains possible that a smaller set of triangular numbers might form a magic square without the corresponding natural numbers forming a magic square. Moreover, it has never been satisfactorily proved that there is no doubly-magic (=bimagic) square of order 7."

By an exhaustive search, we proved with **Walter Trump** in 2002 [2b] that a bimagic square of order smaller or equal to 7 does not exist: this means that Heath's trick in using bimagic squares cannot be used for orders $n < 8$. Here is a study of the problem for the smallest orders.

Order *n*=3, impossible

Is it possible to construct a magic square using the 9 triangular numbers 0, 1, 3, 6, 10, 15, 21, 28, 36?
No! The total sum of these numbers being 120, such a square would have a magic sum 120/3=40. The central number of any 3x3 magic square being one third of its magic sum, and 40 not being divisible by 3, since *n*=3, it is impossible.

Order *n*=4, impossible

A magic square using the 16 triangular numbers 0, 1, 3, …, 120 would have a magic sum equal to 170. There are only 10 series of 4 triangular numbers giving this sum:

0	1	78	91
0	10	55	105
1	21	28	120
1	28	36	105
1	36	55	78
3	10	66	91
3	21	55	91
6	28	45	91
15	28	36	91
21	28	55	66

In a magic square, each number needs to use at least two or three series: one for the row, one for the column, and one more if the number is located on a diagonal. Because the number 6, for example, is present in only one series, a magic square of order *n*=4 is impossible.

Order *n*=5, impossible

A magic square using the 25 triangular numbers 0, 1, 3, …, 300 would have a magic sum equal to 520. There are 118 series giving this sum. Combining the series, there are 148 possible ways to get 5 series using the 25 triangular numbers. This means that it is possible to have 5 magic rows. An exhaustive search, however, shows that it is impossible to arrange these rows and make all the columns magic. The best possible arrangements are 5 magic rows and 3 magic columns, for example:

0	3	105	276	136
66	1	253	190	10
210	45	6	28	231
91	300	36	15	78
153	171	120	21	55

Order n=5. This is not a magic square S=520, the two last columns have sums ≠ S.

Order *n*=6, possible! Solution of the problem.

A magic square using the 36 triangular numbers 0, 1, 3, …, 630 would have a magic sum equal to 1295. There are 1921 series giving this sum.

Good news: it is possible to arrange these series to form magic squares! Here is an example, a solution of our problem.

0	406	120	528	105	136
1	300	435	378	171	10
66	276	496	15	91	351
595	78	153	28	210	231
3	190	55	21	465	561
630	45	36	325	253	6

A solution of Heath's Problem E 496.
Order n=6. Magic square with consecutive triangular numbers from 0 to 630.   S=1295.

Order *n*=7, also possible

The order 7 allows also magic squares of the first triangular numbers.

0	378	1176	210	595	6	435
3	351	45	465	703	1128	105
946	171	561	820	190	21	91
741	528	36	325	120	15	1035
1081	300	55	496	780	10	78
28	666	66	231	276	630	903
1	406	861	253	136	990	153

Order n=7. Magic square with consecutive triangular numbers from 0 to 1176.   S=2800.

Numbers from 1 instead of 0

If we prefer to use consecutive polygonal numbers starting from 1 instead of 0 (see below for da Silva's challenge), a similar reasoning shows that the minimum order is again 6. The 6 series of order 4 and the 91 series of order 5 are not sufficient to construct a magic square. Here are examples of order 6 and 7 starting from 1.

28	666	78	1	528	105
45	276	351	3	406	325
66	378	136	171	190	465
496	21	153	630	15	91
210	10	435	595	36	120
561	55	253	6	231	300

Order n=6. Magic square with consecutive triangular numbers from 1 to 666.   S=1406.

36	406	276	3	528	946	780
45	903	351	6	1225	10	435
561	861	496	741	105	21	190
990	120	630	1	66	703	465
253	136	666	1081	153	91	595
55	378	231	15	820	1176	300
1035	171	325	1128	78	28	210

Order n=7. Magic square with consecutive triangular numbers from 1 to 1225.   S=2975.

Squares of polygonal numbers, $p \leq 10$

We can generalize Heath's Problem E496 to other polygonal numbers.
Reminder: the *i*th *p*-gonal number is equal to

$$((p - 2)i^2 - (p - 4)i)/2.$$

With *p*=3, we get triangular numbers. With *p*=4, we get square numbers. With *p*=5, we get pentagonal numbers. And so on…

Any bimagic square can be used to construct magic squares of $k_2i^2+k_1i+k_0$ numbers: using the same bimagic square of order *n*=8 as R.V. Heath, **Charles W. Trigg** published squares of polygonal numbers for *p*=3, 5, 6, 7, 8 in [9], and for *p*=9, 10 in [10]. But is it possible to construct squares of orders *n* < 8? Yes!

- The case *p*=3, triangular numbers. The smallest solution is *n*=6, as analyzed above.
- The case *p*=4, square numbers. The smallest solution is slightly larger: *n*=7. This question was solved in 2005: I constructed a magic square of squares of order 7 using the first squares 0^2, 1^2, …, 48^2. The magic square is on my website [2c] and is also published in the *MAA MathTrek* column of **Ivars Peterson** [6].
- The cases *p*=5, 6, 7, 8, 9, 10. The smallest solution is again *n*=7 for all these *p*. It might be boring to give all my examples, but here is one with *p*=5, a magic square of pentagonal numbers of order 7.

1617	3015	35	0	1162	715	1520
2882	330	12	5	210	3290	1335
1926	2752	2501	247	117	376	145
70	176	2380	1	3432	925	1080
51	532	2262	2625	1717	287	590
1426	782	22	2035	1001	651	2147
92	477	852	3151	425	1820	1247

Order n=7. Magic square with consecutive pentagonal numbers from 0 to 3432.   S=8064.

An interesting remark: a magic square of polygonal numbers can be turned into a magic square of squares by multiplying each term by $8(p-2)$ then adding $(p-4)^2$ to each term, because:

$$8(p-2)\,[((p-2)i^2 - (p-4)i)/2] + (p-4)^2 = [2(p-2)i^2 - (p-4)]^2$$

An unsolved problem: the smallest magic square of distinct triangular numbers

All the above examples use the first consecutive polygonal numbers. But what is the smallest order *n* if we allow any polygonal numbers, consecutive or not, but distinct?

The first 4x4 magic square of squares, using 16 distinct squares, was constructed by **Euler**, in a letter sent to **Lagrange** in 1770 [3]. I found the first 5x5 magic square of squares in 2004 [2c] [3] [6]. Now I am pleased to give the first 4x4 and 5x5 magic squares of triangular numbers:

66	465	780	91
1	630	105	666
300	171	496	435
1035	136	21	210

Order n=4. Magic square with distinct triangular numbers.   S=1402.

351	0	210	91	171
36	136	153	378	120
105	406	15	231	66
325	253	10	45	190
6	28	435	78	276

Order n=5. Magic square with distinct triangular numbers.   S=823.

It is still unknown if a 3x3 magic square of squares is possible [2d] [3] [6] [7], but what about a 3x3 magic square of triangular numbers? As remarked by **John P. Robertson** (author of [7]), in a private communication of April 2007:

> "If there is a 3x3 magic square of squares, then all the entries are odd, and so congruent to 1 modulo 8. Because if T is a triangular number then $8T + 1$ is a square, and if S is an odd square then $(S - 1)/8$ is a triangular number, the question of whether there is a 3x3 magic square of squares is equivalent to the question of whether there is a 3x3 magic square of triangular numbers."

Open problem. Who will construct a 3x3 magic square of distinct triangular numbers, or its equivalent 3x3 magic square of squares? Or who will prove that it is impossible?

Another unsolved problem: the smallest magic square of distinct pentagonal numbers

We have seen that we do not have the answer to the problem of the smallest magic square of triangular numbers or of square numbers: there are 4x4 magic squares, but we still don't know if 3x3 squares are possible.

But after triangular numbers (*p*=3) and square numbers (*p*=4), what about pentagonal numbers (*p*=5)? We find that 6x6 squares are possible, as shown in the figure below, but it should be possible to construct 5x5 squares, or smaller ones.

1426	1520	1080	176	0	376
1335	5	782	2147	22	287
1	1820	651	1926	145	35
92	51	925	247	2262	1001
1247	852	715	70	532	1162
477	330	425	12	1617	1717

Order n=6. Magic square with distinct pentagonal numbers. S=4578.

Open problem. What are the smallest possible magic squares of distinct pentagonal numbers: 3x3, 4x4, 5x5 or 6x6?

In November 2007, **Lee Morgenstern** worked on this very difficult problem. He constructed the first known 4x4 and 5x5 magic squares of distinct pentagonal numbers. Congratulations!

4030	1001	145	2262	117
70	176	2501	2882	1926
782	3151	1162	425	2035
1426	2147	22	1335	2625
1247	1080	3725	651	852

1426	1247	376	3290	0
4187	35	715	477	925
5	145	3876	51	2262
70	1335	782	1001	3151
651	3577	590	1520	1

Order n=5. Magic squares with distinct pentagonal numbers.
Smallest possible magic sums: S=7555 on the left, S=6333 (if 0 is allowed) on the right.

3725	1908012	659022	20475
760060	115787	500837	1214550
300832	543305	1431305	315792
1526617	24130	70	1040417

12650	1969401	578151	31032
722107	83426	455126	1330575
247051	495650	1557032	291501
1609426	42757	925	938126

Order n=4. Magic squares with distinct pentagonal numbers.
Same smallest possible magic sum: S=2591234.

Lee constructed also this 3x3 semi-magic square:

356972	651	54626
19780	275847	116622
35497	135751	241001

Order n=3. Semi-magic square with distinct pentagonal numbers.
Smallest possible magic sum: S=412249.

All this means that the above open problem becomes now:

Open problem. Who will construct a 3x3 magic square of distinct pentagonal numbers? Or who will prove that it is impossible?

As seen above, a magic square of polygonal numbers can be turned into a magic square of squares: an example of a 3x3 magic square of pentagonal numbers would also solve the 3x3 magic square of squares problem.

Acknowledgements

Particular thanks to **Sebastião A. da Silva** in Brazil, who challenged me to find solutions of order $n < 8$ in March 2007. Without knowing the references [4] [5] to Problem E496 by Heath and [9] [10] to articles by Charles W. Trigg, Sebastião had independently found the relationship with bimagic squares and sent me this solution of order 8, constructed using **G. Pfeffermann**'s first bimagic square built in 1890 [2a].

1596	595	36	1653	171	1128	45	496
561	210	1485	1176	28	435	1770	55
351	946	91	276	2080	741	10	1225
190	15	630	465	1431	78	1081	1830
120	325	2016	3	861	300	1275	820
21	1540	153	66	666	1711	528	1035
1891	136	903	1378	378	1	780	253
990	1953	406	703	105	1326	231	6

Order n=8. Magic square with consecutive triangular numbers from 1 to 2080. S=5720.
Constructed by Sebastião A. da Silva using the Pfeffermann's first bimagic square.

But Sebastião was unsuccessful in finding examples of smaller orders. He offered a bottle of Brazilian Curaçao if I succeeded in answering his question, asking in French:

> "Est-il possible de construire un carré triangulaire quand il n´existe pas un bimagique du même ordre ?"
> ("Is it possible to construct a triangular square when there is no bimagic square of the same order?")

A bottle? Very interesting! It's because I worked on his challenge that I looked for mathematical references and found that the same problem had been proposed already a long time ago in the *Monthly* – without the reward of a bottle – and which had remained unsolved. The only difference is the starting triangular number: Sebastião started from 1, while Heath started from 0. Both cases are solved now. Because I won his challenge, and because it seems unfortunately difficult to send a bottle through the airmail post, Sebastião sent me in May 2007 this nice gift instead of a bottle. Thanks Sebastião for your interesting challenge and for your gift. We will drink together another bottle when you come to Paris or when I go to Rio!

Pão de Açucar, Rio de Janeiro, received from Sebastião A. da Silva

Thanks to **Doug Hensley**, Dept. of Mathematics, Texas A&M University, and **Douglas B. West**, Dept. of Mathematics, University of Illinois at Urbana-Champaign, both in charge now of the "Problems and Solutions" section of *The American Mathematical Monthly*. They were the first to read the solution n=6, after Sebastião, and immediately accepted to publish it, 66 years after the problem was proposed in the same column.

Thanks also to **George P. H. Styan**, McGill University, Montréal, who sent me a copy of the paper [9], which I was unable to find in the main mathematical libraries of Paris. And thanks also to him and to **Evelyn Matheson Styan**, his wife, for correcting my English in this expanded solution of Problem E 496!

References

[1] W. W. R. Ball and H. S. M. Coxeter, *Mathematical Recreations and Essays*, 13th edition, Dover Publications, 1987

[2] C. Boyer, Multimagic squares and cubes web site, www.multimagie.com/indexengl.htm

[2a] www.multimagie.com/English/Bimagic.htm

[2b] www.multimagie.com/English/Smallestbi.htm

[2c] www.multimagie.com/English/SquaresOfSquares4_7.htm

[2d] www.multimagie.com/English/SquaresOfSquares.htm

[3] Christian Boyer, Some notes on the magic squares of squares problem, *The Mathematical Intelligencer* **27** (2005), n°2, 52-64

[4] R. V. Heath, Problem E 496, *The American Mathematical Monthly* **48** (1941), 699

[5] R. V. Heath, A magic square of triangular numbers – Partial solution by the proposer to Problem E 496 [3], *The American Mathematical Monthly* **49** (1942), 476

[6] Ivars Peterson, Magic squares of squares, *MAA Online*, MathTrek column, June 27, 2005, http://www.maa.org/mathland/mathtrek_06_27_05.html

[7] John P. Robertson, Magic squares of squares, *Mathematics Magazine* **69** (1996), n°4, 289-293

[8] H. Schots, Helsch vierkant, *Bulletins de la Classe des Sciences – Académie Royale de Belgique*, 5ème Série **17** (1931), 339-361

[9] Charles W. Trigg, Magic squares with polygonal elements, *School Science and Mathematics* **71** (1971), 195-197

[10] C. W. Trigg, Magic squares with nonagonal and decagonal elements, *Journal of Recreational Mathematics* **5** (1972), 203-204