A search for 3x3 magic squares having more than six square integers among their nine distinct integers.

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> *"Martin LaBar, in The College Mathematics Journal [January 1984, p.69], asked if a 3x3 magic square exists with nine distinct square numbers. (…) Neither such a square nor a proof of impossibility has been found. (…) I here offer $100 to the first person to construct such a square. If it exists, its numbers are sure to be monstrously large."*
>
> Martin Gardner, 1996

Today, eight years after this quotation, nobody has succeeded in winning the $100 of this Gardner's challenge.
However, it is possible to construct a 3x3 square with nine square integers and only one bad magic sum. The smallest example is the following square, first found independently by Lee Sallows and Michael Schweitzer. All the rows and columns, but only one of the two diagonals, have the same magic sum.

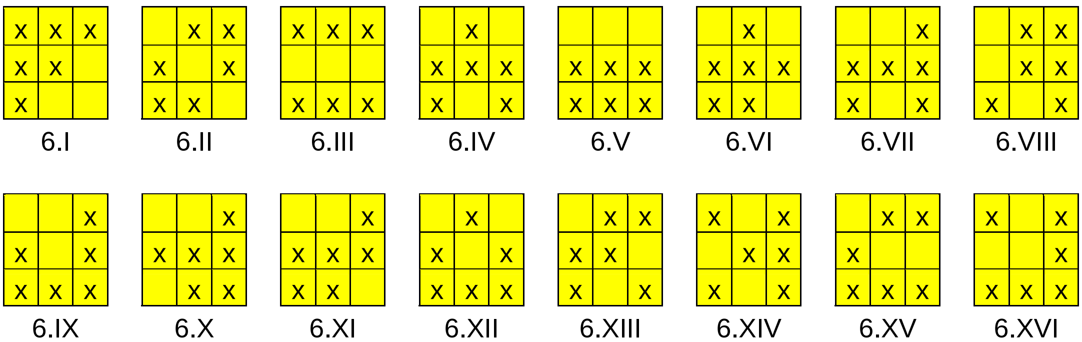
127^2	46^2	58^2
2^2	113^2	94^2
74^2	82^2	97^2

(fig. 1)

Our problem, here, is to get the maximum number of square integers in a fully magic square: the eight lines of the squares of our study will always have the same magic sum.

Six square integers

Andrew Bremner, Department of Mathematics, Arizona State University, demonstrated in 2001 that all the sixteen possible configurations of magic squares including six square integers are possible.



(fig. 2)

Numerous examples with six square integers are easy to find, for each configuration. For example, here is the "smallest" possible magic square with six square integers, "smallest" meaning that it is using the smallest magic sum. This example belongs to the Bremner's configuration 6.XV. The central cell is equal to $145 = 5 \cdot 29$

265	1^2	13^2
7^2	145	241
11^2	17^2	5^2

(fig. 3)

Here are the two smallest examples using a square integer in the central cell. These examples belong to the configurations 6.VII and 6.XIV. There is an easy correspondence between squares of these two configurations, as mentioned by Bremner: that's why these two different squares are in fact very similar, using the same square integers, having one identical diagonal.

889	697	17^2
5^2	25^2	35^2
31^2	553	19^2

5^2	1561	17^2
889	25^2	19^2
31^2	-311	35^2

(fig. 4)

And when two magic squares 3x3 have the same central cell, then they have the same magic sum. The magic sum of a magic square 3x3 always equals three times the central cell.

<u>Seven square integers</u>

Up to symmetry (rotation and reflection), there are eight ways of selecting seven entries from a 3x3 square. These are as follows:

x	x	x
	x	
x	x	x

7.I

x	x	x
x	x	
x		x

7.II

x	x	x
x	x	x
x		

7.III

x	x	x
	x	x
x	x	

7.IV

x	x	x
x	x	x
	x	

7.V

x	x	
x	x	x
	x	x

7.VI

x	x	x
x		x
x		x

7.VII

x	x	x
x		x
x	x	

7.VIII

(fig. 5)

Two results are already known about these configurations:

- Duncan Buell, Department of Computer Science and Engineering, University of South Carolina, studied in 1998 the configuration 7.I, that he called the "magic hourglass", and computed that there is no solution with a central cell $< 25 \cdot 10^{24}$. A direct consequence: if a magic square of squares exists, then its central cell is bigger than $25 \cdot 10^{24}$. Martin Gardner was right saying that *"if it exists, its numbers are sure to be monstrously large."*
- Lee Sallows and Andrew Bremner had separately and previously found the <u>only known example</u> having seven square integers, excluding its symmetries, rotations and k^2 multiples. This example is of configuration 7.IV. The central cell is equal to $425^2 = (5^2 \cdot 17)^2 = 180{,}625$.

373^2	289^2	565^2
360721	425^2	23^2
205^2	527^2	222121

(fig. 6)

The goal of our study is to try to find at least another example with seven square integers.
Of course excluding rotations, symmetries, or k^2 multiples of the figure 6.

Eight square integers

Up to symmetry (rotation and reflection), there are three ways of selecting eight entries from a 3x3 square. These are as follows:

x	x	x
x	x	x
x		x

8.I

x	x	x
x	x	x
x	x	

8.II

x	x	x
x		x
x	x	x

8.III

(fig. 7)

Currently, no example with eight square integers is known (and no example with nine square integers, Martin Gardner's initial challenge).

Some words about the method used

A line going through the central cell C, and having two square integers around the central cell, is an integer solution of the equation:

$$x^2 + y^2 = 2C.$$

Because 4k+3 prime integers cannot be sums of two square integers, we study only magic squares with central cells which are products of 4k+1 prime integers. All the 7.x and 8.x configurations need two, three or four such lines through the centre, meaning at least (as a strict minimum) two, three or four solutions of the above equation.
Knowing that a 4k+1 prime number has only one way to be a sum of two square numbers, knowing that the product $(a^2 + b^2)(c^2 + d^2)$ gives two different ways to be a sum of two square numbers,

- $(ad + bc)^2 + (ac – bd)^2$
- $(ad – bc)^2 + (ac + bd)^2$

and using the fact that

- $2(a^2 + b^2) = (a + b)^2 + (a – b)^2$

it is possible to demonstrate that:

D1. For configurations 7.I to 7.VI, and 8.I to 8.II where the central cell C is a square $C=c^2$.
If c has n distinct factors which are 4k+1 prime integers, then there are:
$(3^n – 1)/2$ different solutions of $x^2 + y^2 = 2c^2$, with $x<y$.

D2. For configurations 7.VII, 7.VIII, 8.III where the central cell C is not a square.
If C has n distinct factors which are 4k+1 prime integers, then there are:
$2^{(n-1)}$ different solutions of $x^2 + y^2 = 2C$, with $x<y$.

Even if distinct factors give the above maximum number of solutions, it is interesting for variety purpose, to allow factors in common in the factorisation of the central cell: the only known example with 7 squares (fig.6) has one time the factor 17, but twice the factor 5.

Research with a square integer in the central cell

We first limit our study to magic squares having a square integer in the central cell: all the configurations from 7.I to 7.VI, and 8.I, 8.II.

Disappointing result, done by computer: **the magic square of Fig 6 is the ONLY magic square** (excluding its rotations, symmetries, and k^2 multiples) **with more than six square integers, if the central cell is** one of the following square integer types:

a) $(5^i \cdot p_1 \cdot p_2)^2$ with $5 \le p_j < 40{,}000$ ($\rightarrow$ central cell $< 1.60 \times 10^{21}$)
b) $(5^i \cdot p_1 \cdot p_2 \cdot p_3)^2$ with $5 \le p_j < 1{,}500$ ($\rightarrow$ central cell $< 6.92 \times 10^{21}$)
c) $(5^i \cdot p_1 \cdot p_2 \cdot p_3 \cdot p_4)^2$ with $5 \le p_j < 300$ ($\rightarrow$ central cell $< 3.39 \times 10^{22}$)
d) $(5^i \cdot p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5)^2$ with $5 \le p_j \le 101$ ($\rightarrow$ central cell $< 6.90 \times 10^{22}$)
e) $(5^i \cdot p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5 \cdot p_6)^2$ with $5 \le p_j \le 53$ ($\rightarrow$ central cell $< 3.07 \times 10^{23}$)
f) $(5^i \cdot p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5 \cdot p_6 \cdot p_7)^2$ with $5 \le p_j \le 37$ ($\rightarrow$ central cell $< 5.63 \times 10^{24}$)
g) $(5^i \cdot p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5 \cdot p_6 \cdot p_7 \cdot p_8)^2$ with $5 \le p_j \le 29$ ($\rightarrow$ central cell $< 1.56 \times 10^{26}$)
h) $(5^i \cdot p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5 \cdot p_6 \cdot p_7 \cdot p_8 \cdot p_9 \cdot p_{10})^2$ with $5 \le p_j \le 17$ ($\rightarrow$ central cell $< 2.54 \times 10^{27}$)
i) $(5^i \cdot p_1 \cdot p_2 \cdot p_3)^2$ with
 $5 \le p_1 \le 101$
 $5 \le p_2 < 1{,}000$
 $5 \le p_3 < 10{,}000$ ($\rightarrow$ central cell $< 6.30 \times 10^{20}$)
j) $(5^i \cdot p_1 \cdot p_2 \cdot p_3 \cdot p_4)^2$ with
 $5 \le p_1 \le 101$
 $5 \le p_2 < 150$
 $5 \le p_3 < 300$
 $5 \le p_4 < 1{,}000$ ($\rightarrow$ central cell $< 1.21 \times 10^{22}$)
k) $(5^i \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot p_5 \cdot p_6)^2$ with
 $5 \le p_5 \le 41$
 $5 \le p_6 < 1{,}000$ ($\rightarrow$ central cell $< 5.87 \times 10^{22}$)
l) $(5^{i_0} \cdot (p_1)^{i_1} \cdot (p_2)^{i_2} \cdot (p_3)^{i_3})^2$ with $5 \le p_j < 200$ ($\rightarrow$ central cell $< 2.14 \times 10^{30}$)
m) $(p_1 \cdot p_2 \cdot p_3)^2$ with $5 \le p_j < 3{,}000$ ($\rightarrow$ central cell $< 6.85 \times 10^{20}$)

$0 \le i \le 2$, and p_j being a 4k+1 prime number {5, 13, 17, 29, 37, 41, 53, …}

From a) to k), I have in fact analyzed only $(5^2 \cdot p_1 \cdot \ldots)^2$. And for l), only $(5^2 \cdot p_1^2 \cdot p_2^2 \cdot p_3^2)^2$.
Because a magic square with x square entries keeps its x square entries when all the cells are multiplied by the same square factor, our results includes automatically all the submultiples of the square root of the central cell: if there is no magic square with central cells $(5^2 \cdot p_1 \cdot \ldots)^2$, then there is also no magic square with central cells $(5 \cdot p_1 \cdot \ldots)^2$ and $(p_1 \cdot \ldots)^2$ in the studied intervals. And also, for example, the type a) includes automatically the simplest case $(5^i \cdot p_1)^2$.

Research with a non-square integer in the central cell

We now study the magic squares that do not have a square integer in the central cell: configurations 7.VII, 7.VIII, and 8.III.

Again a disappointing result, done by computer: **there is NO magic square with more than six square integers, if the central cell is** one of the following integer types:

n) $(5^i \cdot p_1)$ with $5 \le p_j < 40{,}000$ ($\rightarrow$ central cell $< 1.25 \times 10^{8}$)
o) $(5^i \cdot p_1 \cdot p_2)$ with $5 \le p_j < 40{,}000$ ($\rightarrow$ central cell $< 5 \times 10^{12}$)
p) $(5^i \cdot p_1 \cdot p_2 \cdot p_3)$ with $5 \le p_j < 4{,}000$ ($\rightarrow$ central cell $< 1.98 \times 10^{14}$)
q) $(5^i \cdot p_1 \cdot p_2 \cdot p_3 \cdot p_4)$ with $5 \le p_j < 800$ ($\rightarrow$ central cell $< 1.26 \times 10^{15}$)

r) $(5^i \cdot p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5)$ with $5 \le p_j < 300$ (→ central cell < 6.75×10^{15})
s) $(5^i \cdot p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5 \cdot p_6)$ with $5 \le p_j < 150$ (→ central cell < 3.42×10^{16})
t) $(5^i \cdot p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5 \cdot p_6 \cdot p_7)$ with $5 \le p_j \le 101$ (→ central cell < 3.35×10^{17})
u) $(5^i \cdot p_1 \cdot p_2 \cdot p_3)$ with
  $5 \le p_1 < 150$
  $5 \le p_2 < 1{,}500$
  $5 \le p_3 < 15{,}000$ (→ central cell < 1.04×10^{13})
v) $(5^i \cdot p_1 \cdot p_2 \cdot p_3 \cdot p_4)$ with
  $5 \le p_1 < 150$
  $5 \le p_2 < 250$
  $5 \le p_3 < 500$
  $5 \le p_4 < 1{,}500$ (→ central cell < 7.72×10^{13})
w) $(5^i \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot p_5 \cdot p_6)$ with
  $5 \le p_5 \le 101$
  $5 \le p_6 < 1{,}500$ (→ central cell < 1.12×10^{14})
x) $(p_1 \cdot p_2 \cdot p_3)$ with $5 \le p_j < 15{,}000$ (→ central cell < 3.35×10^{12})

$0 \le i \le 5$, and p_j being a 4k+1 prime number {5, 13, 17, 29, 37, 41, 53, …}

I have in fact analyzed only $(5^4 \cdot p_1 \cdot \ldots)$ and $(5^5 \cdot p_1 \cdot \ldots)$ integers.
About $(5^4 \cdot p_1 \cdot \ldots)$: because the square properties of a magic square are not modified by a square factor like 5^2, if there is no magic square with central cell $(5^4 \cdot p_1 \cdot \ldots)$, then there is also no magic square with central cells $(5^2 \cdot p_1 \cdot \ldots)$ and $(p_1 \cdot \ldots)$ in the studied intervals.
About $(5^5 \cdot p_1 \cdot \ldots)$: because the square properties of a magic square are not modified by a square factor like 5^2, if there is no magic square with central cell $(5^5 \cdot p_1 \cdot \ldots)$, then there is also no magic square with central cells $(5^3 \cdot p_1 \cdot \ldots)$ and $(5 \cdot p_1 \cdot \ldots)$ in the studied intervals.

Conclusion

Fig 6 shows, by the example, that it is possible to get seven square entries in a magic square. It is very strange (and really disappointing…) to have been unable to find at least another example of such a square in our various and wide range of central cells.
It is so difficult to find at least another example with “only” seven squares that I think that a complete square of squares -with nine square integers- cannot exist. But it’s only a feeling, the eventual proof of the impossibility has not yet been found…

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