3 x 3 Near-Magic Squares of Squares

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If a square fails to be magic because the total on only one diagonal is different to the row and column totals, it is said to be a near-magic square. Our objective in this note is to construct a parameterisation that allows us to systematically generate such an array, but with entries that consist of square numbers only. However, before we can continue we require some preliminary results.

We initially note that if a 3×3 magic square contains distinct integer squares, there exist natural numbers *a*, *u* and *v* (possibly after a reflection or rotation) such that the square can be written in the form of the leftmost square in figure 1. Furthermore, if we rearrange the terms into the form given in the rightmost square in figure 1, we see that the rows each form arithmetic progressions with common difference *u* and the columns each form arithmetic progressions with common difference *v* [2], [5].

$a+u+2v$	a	$a+2u+v$
$a+2u$	$a+u+v$	$a+2v$
$a+v$	$a+2u+2v$	$a+u$

$\rightarrow$

a	$a+u$	$a+2u$
$a+v$	$a+u+v$	$a+2u+v$
$a+2v$	$a+u+2v$	$a+2u+2v$

Figure 1: Constructing arithmetic progressions from the general form of a 3×3 magic square

This suggests that we need to find a method that allows us to systematically find sets of three squares that are in arithmetic progression and then use these to reconstruct the original square. We initially note that if x^2, y^2 and z^2 are three such squares, then $y^2-x^2=z^2-y^2$. Hence $x^2+z^2=2y^2$, which allows us to make use of the following well known Theorem [1, pp.435-440], [4, pp.314-315]:

If (x,y,z) is a primitive solution of the equation $x^2+z^2=2y^2$ in natural numbers, there exist relatively prime integers *r* and *s* of opposite parity with $0<r<s$ such that $x=|r^2-s^2-2rs|$, $y=r^2+s^2$ and $z=r^2-s^2+2rs$.

In fact, all arithmetic progressions of three squares are either of this form or integer multiples of it. So, in order to proceed further, we must reverse this result and find suitable values of *r* and *s* for which $x^2+z^2=2y^2$. Fortunately, Dickson [1, p.174] reports that C.E. Hillyer gave the following three transformations back in 1900 that will always generate values of *x*, *y* and *z* with the desired property:

$$(r,s)=(m^2+mn+n^2,m^2-n^2)=(m^2+mn+n^2,2mn+n^2)=(m^2+2mn,m^2+mn+n^2)$$

We will now show how to apply these combinations to construct a near-magic square of square numbers that fails only on the diagonal from the bottom left of the array to the top right. Accordingly, we have the following algorithm:

An algorithm to generate 3 x 3 near-magic squares of squares

Let (m,n) be an arbitrary pair of distinct natural numbers. Then, for $i=1,2,3$,

- Evaluate $(r_1,s_1)=(m^2+mn+n^2, m^2-n^2)$, $(r_2,s_2)=(m^2+mn+n^2, 2mn+n^2)$ and $(r_3,s_3)=(m^2+2mn, m^2+mn+n^2)$.
- Find $x_i=|r_i^2-s_i^2-2r_is_i|$, $y=r_i^2+s_i^2$ and $z=r_i^2-s_i^2+2r_is_i$ for each (r_i,s_i) pair computed above, and hence obtain three sets of integers (x_i,y_i,z_i) whose squares are in arithmetic progression.
- Obtain the near-magic square of squares by the mapping $x_1^2\to a_{12}$, $y_1^2\to a_{33}$, $z_1^2\to a_{21}$, $x_2^2\to a_{31}$, $y_2^2\to a_{22}$, $z_2^2\to a_{13}$, $x_3^2\to a_{23}$, $y_3^2\to a_{11}$ and $z_3^2\to a_{32}$.

We will illustrate this algorithm with the parameters $m=2$ and $n=1$, and it follows that $(r_1,s_1)=(7,3)$, $(r_2,s_2)=(7,5)$ and $(r_3,s_3)=(8,7)$. Hence $(x_1,y_1,z_1)=(2,58,82)$, $(x_2,y_2,z_2)=(46,74,94)$, $(x_3,y_3,z_3)=(97,113,127)$ and the final mapping gives the near-magic square displayed in figure 2.

113^2	2^2	94^2
82^2	74^2	97^2
46^2	127^2	58^2

Figure 2: The 3×3 near-magic square that is output from the above algorithm when $m=2$ and $n=1$

This is actually the smallest square in this particular family with row, column and leading diagonal totals of 147^2. Had we have used the parameters $m=1$ and $n=2$ instead, the entries in the square would be the same but occupying different positions. And although the magic sum is still 147^2, the total on the rogue diagonal would now be 38307 as opposed to 16428.

If we run the algorithm without assigning numerical values to *m* and *n* we arrive at the parametric family of near-magic squares given in figure 3. The sum of the rows, columns and leading diagonal are each $9(m^2+mn+n^2)^4$, and the sum of the terms on the truant diagonal is $3(m^4+2m^3n+7m^2n^2+6mn^3+2n^4)^2$.

$\left[(m^2+2mn)^2+(m^2+mn+n^2)^2\right]^2$	$(2n^4+5m^2n^2+4mn^3-2m^4)^2$	$(m^4+6m^3n+5m^2n^2+4mn^3+2n^4)^2$
$(2m^4+4m^3n+5m^2n^2-2n^4)^2$	$\left[(m^2+mn+n^2)^2+(2mn+n^2)^2\right]^2$	$(2m^4+4m^3n+5m^2n^2+6mn^3+n^4)^2$
$(2n^4+2m^3n+7m^2n^2+8mn^3-m^4)^2$	$(2m^4+8m^3n+7m^2n^2+2mn^3-n^4)^2$	$\left[(m^2-n^2)^2+(m^2+mn+n^2)^2\right]^2$

Figure 3: A parametric family of 3×3 near-magic squares

It would, of course, be very nice to equate the diagonal sums in an attempt to find values for *m* and *n* which satisfied both. In other words, look for an integer solution to the equation $9(m^2+mn+n^2)^4 = 3(m^4+2m^3n+7m^2n^2+6mn^3+2n^4)^2$.

However, there is little point as the magic sum of any 3×3 magic square is 3 times the a_{22} entry. It follows that the magic sum of a 3×3 magic square of squares is $3k^2$ for some natural number *k*, and our square is not of this form. In other words, should a 3×3 magic square of squares exist it will not be a member of this family.

Another interesting observation is that this family is actually a sub-family of Lucas' parametric family of semi-magic squares given in figure 4.

$(p^2+q^2-r^2-s^2)^2$	$[2(qr+ps)]^2$	$[2(qs-pr)]^2$
$[2(qr-ps)]^2$	$(p^2-q^2+r^2-s^2)^2$	$[2(rs+pq)]^2$
$[2(qs+pr)]^2$	$[2(rs-pq)]^2$	$(p^2-q^2-r^2+s^2)^2$

Figure 4: Lucas' parametric family of 3×3 semi-magic squares of perfect squares

A transformation (one of many) that will achieve this mapping is given by

$$p \to \sqrt{2}(m^2+mn+n^2),\; q \to -\frac{1}{\sqrt{2}}m(m+2n),\; r \to -\frac{1}{\sqrt{2}}n(2m+n),\; s \to \frac{1}{\sqrt{2}}(n^2-m^2)$$

So, for example, substituting $p = 7\sqrt{2}$, $q = -4\sqrt{2}$, $r = -5/\sqrt{2}$ and $s = -3/\sqrt{2}$ into the Lucas family gives the 3×3 near-magic square illustrated in figure 2.

It is, of course, a simple matter to reduce the parameterisation given in figure 3 to a single variable family, and figure 5 shows the result of applying the transformations $m \to k+1$ and $n \to k$. This sub-family has row, column and leading diagonal totals of $9(1+3k+3k^2)^4$, and reduces to the square in figure 2 if we take $k=1$.

$(2+14k+37k^2+42k^3+18k^4)^2$	$(-2-8k-7k^2+6k^3+9k^4)^2$	$(1+10k+29k^2+36k^3+18k^4)^2$
$(2+12k+29k^2+30k^3+9k^4)^2$	$(1+6k+19k^2+30k^3+18k^4)^2$	$(2+12k+29k^2+36k^3+18k^4)^2$
$(-1-2k+7k^2+24k^3+18k^4)^2$	$(2+16k+43k^2+48k^3+18k^4)^2$	$(2+10k+19k^2+18k^3+9k^4)^2$

Figure 5: A family of 3×3 near-magic squares of squares, defined in terms of a single variable

We finally note that not all near-magic squares can be generated from our original parameterisation, and the example displayed in figure 6 falls into this category. This was discovered by Michael Schweitzer [3] and its row, column and (non-leading) diagonal each sum to 20966014. As this is not a perfect square it cannot be output from our algorithm.

35^2	3495^2	2958^2
3642^2	2125^2	1785^2
2775^2	2058^2	3005^2

Figure 6: Michael Schweitzer's 3×3 near-magic squares of squares

References

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