September 1, 2012

THE MINIMUM ORDER FOR

AN AXIALLY SYMMETRIC BIMAGIC SQUARE IS 12

We call “axially symmetric” a square where the pairs of complementary numbers are located in cells which are symmetric to a vertical (or to an horizontal) axis.

Examples for the magic squares of the order 4:

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01  08  12  13      or      01  11  06  16
11  14  02  07              08  14  03  09
06  03  15  10              12  02  15  05
16  09  05  04              13  07  10  04
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It is the Dudeney’s type VI.

The question here is to find the minimum order of an axially symmetric bimagic square. Naturally, the order of such a square should be even.

1. THERE ARE NO AXIALLY SYMMETRIC BIMAGIC 8x8 SQUARES (by F. Gaspalou)

I search an axially symmetric bimagic 8x8 square.

I can try to build an auxiliary 8x8 square where
- the columns are magic (sum=260) and bimagic (sum of squares=11180) ; but the 2 main diagonals are not necessarily magic nor bimagic
- the numbers in the columns are ordered (A1<B1<C1<D1
A2<B2<C2<D2,
…
and E1=65-D1, F1=65-C1, G1=65-B1, H1=65-A1
….

I have then to distribute the 32 first numbers in the 4 first rows, with these conditions.

I made a program for that. It is a program with 7*4=28 parameters:
A1, B1, C1, D1,
A2 B2, C2, D2,
…
A7, B7, C7, D7
It is a backtracking program with 28 imbricated loops.

In fact, this program gives 0 solution:

I can find solution only for the first 27 loops. Example:

01	02	03	04	06	08	05	x
14	12	13	15	11	09	07	x
27	29	24	17	22	20	10	x
32	31	26	30	23	25	x	x

but it is impossible to find a 28^{th} parameter giving a solution.

Conclusion: it is impossible to build an axially symmetric semi-bimagic 8x8 square
and a fortiori an axially symmetric bimagic 8x8 square.

Idem for the similar types using 4 complementary pairs of numbers in each column (or row), like for example the type

A1+B1=65, C1+D1=65, E1+F1=65, G1+H1=65

My proof uses a program and it is maybe possible to find a direct demonstration.

An **other way of proof** is to consider the 81 bimagic series made with 4 complementary pairs of numbers:

1^{st} 01 14 27 32 33 38 51 64
2^{nd} 01 17 21 32 33 44 48 64
….
81th 11 12 16 18 47 49 53 54

We have to find 8 bimagic series (out of 81) and the 4 first numbers of these series have to use exactly all the different numbers from 1 to 32. I am afraid the only way of proof there is no solution (or the easier way at least to prove it) is also to build a program for that.

2. MINIMUM ORDER (by W. Trump)

Bimagic squares do not exist for orders smaller than 8.
See http://www.multimagie.com/English/Smallestbi.htm on the website of Christian Boyer.

Order m = 8

See chapter 1. The proof of Francis Gaspalou could be confirmed by the suggested examination of the 81 symmetric bimagic series for squares of order 8 with a computer program. There are 4993 combinations of 6 distinct series. But a combination of more than 6 distinct series does not exist.
There simply are not enough symmetric bimagic series of order 8.

Order m = 10

Symmetric series of order 10 consist of exactly 5 even and 5 odd numbers as $(m^2+1) = 101$. The square of an even number is equivalent to 0 modulo 4 and the square of an odd number is

equivalent to 1 modulo 4. Therefore the sum of the squares of the numbers of any symmetric series of order 10 is equivalent to $(5\cdot 0 + 5\cdot 1) \equiv 1$ modulo 4. But for $m = 10$ the magic sum of the squares of the numbers is 33835 which is equivalent to 3 modulo 4. Therefore no symmetric series of order 10 is bimagic.
(General investigations show that there are no symmetric series for $m \equiv 2 \bmod 4$.)

Order m = 12

Since 2002 an axially symmetric trimagic square of order 12 is known. See http://www.multimagie.com/English/Trimagic12.htm on the website of Christian Boyer. Of course this square is bimagic.
Thus 12 is the smallest order for which axially symmetric bimagic squares exist.