What are the *smallest* possible magic squares?

Twelve enigmas for winning €8,000 and twelve bottles of champagne!

(€8,000 ≈ $10,000 at the time of this document)

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While magic squares have been known and studied for many centuries, it is surprising that for certain types of magic squares we still do not know today which are the smallest possible! For example, even though Euler sent this 4x4 magic square of squares to Lagrange as early as 1770:

68^2	29^2	41^2	37^2
17^2	31^2	79^2	32^2
59^2	28^2	23^2	61^2
11^2	77^2	8^2	49^2

A 4x4 magic square of squares by Euler.
An nxn magic square uses n^2 distinct integers and has the same sum S for its n rows, its n columns and its 2 diagonals. Here S = 8,515.

we still do not know if a 3x3 magic square of squares is possible!

a^2	b^2	c^2
d^2	e^2	f^2
g^2	h^2	i^2

Nobody has been able to build a 3x3 magic square with 9 distinct squared integers.

In an effort to make progress on these unsolved problems, twelve prizes totaling €8,000 and twelve bottles of champagne are offered for the solutions to twelve enigmas. The prizes are as follows:

- Six main enigmas, €1,000 each, totaling €6,000 and 6 bottles
- Six small enigmas, €100 to €500 each, totaling €2,000 and 6 bottles

Of course, only the first person who solves an enigma will win the corresponding prize. Here is a summary of the enigmas, with more detail on the following pages. For each enigma, you must produce an example or prove it impossible. The small enigmas are in parentheses.

1. 3×3 magic square using at least seven squared integers, different from the only known example.
2. 5x5 bimagic square.
3. 3x3 (7x7) semi-magic square of cubes.
4. 4x4 (5x5, 6x6, 7x7) magic square of cubes.
5. Multiplicative magic cube using integers < 364.
6. 5x5 (6x6, 7x7) additive-multiplicative magic square.

These enigmas can be mathematically rewritten as sets of Diophantine equations: for example, a 3x3 magic square is a set of 8 equations in 10 unknowns (the 10^{th} unknown being the magic constant to which each line must sum).

Each enigma will allow the table below to be completed with the name of the first person to have solved it (thus winning the prize) by either building such a square or proving it impossible.

	Magic squares of squares	Bimagic squares	Semi-magic squares of cubes	Magic squares of cubes	Add-mult magic squares
2x2	Impossible				
3x3	**Main enigma #1 (€1000)***	Impossible. Proved by E Lucas, 1891	**Main enigma #3 (€1000)**	Impossible	Impossible. Proved by L. Morgenstern, 2007
4x4	L. Euler, 1770	Impossible. Proved by L. Pebody / J.-C. Rosa**, 2004	L. Morgenstern, 2006	**Main enigma #4 (€1000)**	
5x5	C. Boyer, 2004	**Main enigma #2 (€1000)**	C. Boyer, 2004	**Small enigma #4a (€500)**	**Main enigma #6 (€1000)**
6x6	C. Boyer, 2005	J. Wroblewski, 2006	L. Morgenstern, 2006	**Small enigma #4b (€500)**	**Small enigma #6a (€500)**
7x7	C. Boyer***, 2005	L. Morgenstern, 2006	**Small enigma #3a (€100)**	**Small enigma #4c (€200)**	**Small enigma #6b (€200)**
8x8	G. Pfeffermann***, 1890		L. Morgenstern, 2006	W. Trump, 2008	W. Horner, 1955
9x9	G. Pfeffermann***, 1891		L. Morgenstern - C. Boyer, 2006	C. Boyer***, 2006	W. Horner, 1952

* or using at least 7 squared integers on its 9 integers, different from the only known example

** proved the same year, but independently

*** these squares use consecutive integers (or consecutive squared integers, or consecutive cubed integers)

Countries: Switzerland (Euler), England (Pebody), France (Pfeffermann, Lucas, Rosa, Boyer), Germany (Trump), Poland (Wroblewski), USA (Horner, Morgenstern)

Recapitulative table of enigmas and of first discoverers.
The main enigma #5 does not appear here: different, it is the only one concerning magic cubes.

The solutions to the enigmas must be sent to Christian Boyer, cboyer@club-internet.fr.
The www.multimagie.com/indexengl.htm website gives more information about every enigma, and will contain regular updates regarding received progress and prizes won.

Before these prizes adding up €8,000 are offered, there were publications by Christian Boyer on these enigmas. In chronological order:

This *www.multimagie.com* website, open since 2002, details in French, English and German the first results of searchers on these problems, as well as on many other problems concerning magic squares, cubes and hypercubes.

In 2005, three main enigmas (#1, #2, #4) were a part of open problems of the paper "Some Notes on the Magic Squares of Squares Problem" published in *The Mathematical Intelligencer*, Vol. 27, number 2.

In November 2007, main enigma #5 on multiplicative magic cubes was submitted to the readers of the mathematical French magazine *Tangente* number 119.

In April 2008, in the paper "Enigmes sur les Carrés Magiques" published in *Dossier Pour La Science* number 59 "Jeux Math", (*Pour La Science* is the French edition of *Scientific American*) €100 were offered for each of the first five main enigmas. Now, two years later, a sum ten times as large is offered: €1,000 each.

Again in 2008, main enigma #4 was an unsolved problem mentioned in part 8.3 of the paper "New Upper Bounds for Taxicab and Cabtaxi Numbers" published in *The Journal of Integer Sequences*, Vol. 11, number 1.

In April-May 2009, the *www.pourlascience.fr* website published the first five main enigmas in its "Games" column ("Jeux" in French). Main enigma #6 on add-mult squares was then added in June 2009.

Before submitting these enigmas, Christian Boyer solved some problems on magic squares and magic cubes:

In 2001, he constructed with André Viricel the first known pentamagic square and published it in *Pour La Science*. This magic square stays magic after squaring its integers, stays magic after cubing its integers, stays magic after raising at the 4th power its integers, and stays magic after raising at the 5th power its integers.

In 2003, he solved with Walter Trump the old problem of the smallest possible perfect magic cube, and published it in *La Recherche*. Numerous magazines all around the world published their cube. The problem had been popularized by Martin Gardner in *Scientific American* in 1976, and initially studied by Pierre de Fermat as early as 1640.

In 2007, he solved the old problem of the smallest possible magic square of triangular numbers, and published the solution in *The American Mathematical Monthly*. This problem had been initially posed 66 years earlier by Royal V. Heath, in 1941, also in *The American Mathematical Monthly*.

“What are the smallest…?”
Detail of the 12 enigmas.

What are the smallest possible magic squares of squares: 3x3 or 4x4?

In 1770 Leonhard Euler was the first to construct 4x4 magic squares of squares, as mentioned above. But nobody has yet succeeded in building a 3x3 magic square of squares or proving that it is impossible. Edouard Lucas worked on the subject in 1876. Then, in 1996, Martin Gardner offered $100 to the first person who could build one. Since this problem – despite its very simple appearance – is incredibly difficult to solve with nine distinct squared integers, here is an enigma which should be easier:

- **Main enigma #1 (€1000 and 1 bottle)**. Construct a 3x3 magic square using seven (or eight, or nine) distinct squared integers different from the only known example and of its rotations, symmetries and k^2 multiples. Or prove that it is impossible.

373^2	289^2	565^2
360721	425^2	23^2
205^2	527^2	222121

Only known example of 3x3 magic square using seven distinct squared integers, by Andrew Bremner. S = 541,875.

What are the smallest possible bimagic squares: 5x5 or 6x6?

A bimagic square is a magic square which stays magic after squaring its integers. The first known were constructed by the Frenchman G. Pfeffermann in 1890 (8x8) and 1891 (9x9). The 3x3 and 4x4 bimagics have been mathematically proven impossible. The smallest bimagics currently known are 6x6, the first one of which was built in 2006 by Jaroslaw Wroblewski, a mathematician at Wroclaw University, Poland.

17	36	55	124	62	114
58	40	129	50	111	20
108	135	34	44	38	49
87	98	92	102	1	28
116	25	86	7	96	78
22	74	12	81	100	119

A 6x6 bimagic square by Jaroslaw Wroblewski. S1 = 408, S2 = 36,826.

- **Main enigma #2 (€1000 and 1 bottle)**. Construct a 5x5 bimagic square using distinct positive integers. Or prove that it is impossible.

What are the smallest possible semi-magic squares of cubes: 3x3 or 4x4?

An nxn semi-magic square is a square whose n lines and n columns have the same sum, but whose diagonals can have any sum. The smallest semi-magic squares of cubes currently known are 4×4 constructed in 2006 by Lee Morgenstern, an American mathematician. We also know 5x5 and 6x6 squares, then 8x8 and 9x9, but not yet 7x7.

16^3	20^3	18^3	192^3
180^3	81^3	90^3	15^3
108^3	135^3	150^3	9^3
2^3	160^3	144^3	24^3

A 4x4 semi-magic square of cubes by Lee Morgenstern. S = 7,095,816.

- **Main enigma #3 (€1000 and 1 bottle)**. Construct a 3x3 semi-magic square using positive distinct cubed integers. Or prove that it is impossible.

- **Small enigma #3a (€100 and 1 bottle)**. Construct a 7x7 semi-magic square using positive distinct cubed integers. Or prove that it is impossible.

What are the smallest possible magic squares of cubes: 4x4, 5x5, 6x6, 7x7 or 8x8?

The first known magic square of cubes was constructed by the Frenchman Gaston Tarry in 1905, thanks to a large 128x128 trimagic square (magic up to the third power). The smallest currently known magic squares of cubes are 8x8 squares constructed in 2008 by Walter Trump, a German teacher of mathematics. We do not know any 4x4, 5x5, 6x6 or 7x7 squares. The 3x3 are proven impossible.

11^3	9^3	15^3	61^3	18^3	40^3	27^3	68^3
21^3	34^3	64^3	57^3	32^3	24^3	45^3	14^3
38^3	3^3	58^3	8^3	66^3	2^3	46^3	10^3
63^3	31^3	41^3	30^3	13^3	42^3	39^3	50^3
37^3	51^3	12^3	6^3	54^3	65^3	23^3	19^3
47^3	36^3	43^3	33^3	29^3	59^3	52^3	4^3
55^3	53^3	20^3	49^3	25^3	16^3	5^3	56^3
1^3	62^3	26^3	35^3	48^3	7^3	60^3	22^3

An 8x8 magic square of cubes by Walter Trump. S = 636,363.

- **Main enigma #4 (€1000 and 1 bottle)**. Construct a 4x4 magic square using distinct positive cubed integers. Or prove that it is impossible.

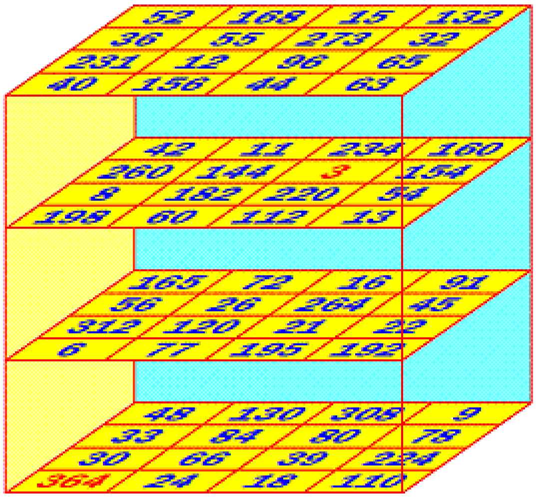
- **Small enigma #4a (€500 and 1 bottle)**. Construct a 5x5 magic square using distinct positive cubed integers. Or prove that it is impossible.

- **Small enigma #4b (€500 and 1 bottle)**. Construct a 6x6 magic square using distinct positive cubed integers. Or prove that it is impossible.

- **Small enigma #4c (€200 and 1 bottle)**. Construct a 7x7 magic square using distinct positive cubed integers. Or prove that it is impossible. (*When such a square is constructed, if small enigma #3a of the 7x7 semi-magic is not yet solved, then the person will win both prizes – that is to say a total of €300 and 2 bottles.*)

What are the smallest integers allowing to construct a multiplicative magic cube?

Contrary to all other enigmas which concern the magic squares, this one concerns the magic cubes. An nxnxn multiplicative magic cube is a cube in which its n^2 rows, n^2 columns, n^2 pillars, and 4 main diagonals have the same product P. Today the best multiplicative magic cubes are 4x4x4 cubes in which the largest used number among their 64 integers is equal to 364. We do not know if it is possible to construct a cube with smaller numbers.



A 4x4x4 multiplicative magic cube by Christian Boyer. Max number = 364. P = 17,297,280.

- **Main enigma #5 (€1000 and 1 bottle)**. Construct a multiplicative magic cube in which the distinct positive integers are all strictly lower than 364. The size is free: 3x3x3, 4x4x4, 5x5x5,…. Or prove that it is impossible.

What are the smallest possible additive-multiplicative magic squares: 5x5, 6x6, 7x7 or 8x8?

An nxn additive-multiplicative magic square is a square in which its n rows, n columns and 2 diagonals have the same sum S, and also the same product P. The smallest known are 8x8 squares, the first one of which was constructed in 1955 by Walter Horner, an American teacher of mathematics. We do not know any 5x5, 6x6 or 7x7 squares. The 3x3 and 4x4 are proven impossible.

162	207	51	26	133	120	116	25
105	152	100	29	138	243	39	34
92	27	91	136	45	38	150	261
57	30	174	225	108	23	119	104
58	75	171	90	17	52	216	161
13	68	184	189	50	87	135	114
200	203	15	76	117	102	46	81
153	78	54	69	232	175	19	60

An 8x8 additive-multiplicative magic square by Walter Horner.
S = 840, P = 2,058,068,231,856,000.

- **Main enigma #6 (€1000 and 1 bottle)**. Construct a 5x5 additive-multiplicative magic square using distinct positive integers. Or prove that it is impossible.

- **Small enigma #6a (€500 and 1 bottle)**. Construct a 6x6 additive-multiplicative magic square using distinct positive integers. Or prove that it is impossible.

- **Small enigma #6b (€200 and 1 bottle)**. Construct a 7x7 additive-multiplicative magic square using distinct positive integers. Or prove that it is impossible.

Contact: Christian Boyer, cboyer@club-internet.fr, France.