GastonTarry: Pandiagonal bimagic Squares

In 1903, Gaston Tarry published some patterns[1] to construct pandiagonal bimagic squares of order 8 with trimagic diagonals.

a $p+r$	$b-c$ $q-r+s$	$b\ \ +d$ p	$a+c+d$ $q\ \ +s$	b $p+r+s$	$a+c$ $q-r$	$a\ \ +d$ $p\ \ +s$	$b-c+d$ q
b p	$a+c$ $q\ \ +s$	$a\ \ +d$ $p+r$	$b-c+d$ $q-r+s$	a $p\ \ +s$	$b-c$ q	$b\ \ +d$ $p+r+s$	$a+c+d$ $q-r$
$a+c+d$ $p+r+s$	$b\ \ +d$ $q-r$	$b-c$ $p\ \ +s$	a q	$b-c+d$ $p+r$	$a\ \ +d$ $q-r+s$	$a+c$ p	b $q\ \ +s$
$b-c+d$ $p\ \ +s$	$a\ \ +d$ q	$a+c$ $p+r+s$	b $q-r$	$a+c+d$ p	$b\ \ +d$ $q\ \ +s$	$b-c$ $p+r$	a $q-r+s$
$a\ \ +d$ $q-r$	$b-c+d$ $p+r+s$	b q	$a+c$ $p\ \ +s$	$b\ \ +d$ $q-r+s$	$a+c+d$ $p+r$	a $q\ \ +s$	$b-c$ p
$b\ \ +d$ q	$a+c+d$ $p\ \ +s$	a $q-r$	$b-c$ $p+r+s$	$a\ \ +d$ $q\ \ +s$	$b-c+d$ p	b $q-r+s$	$a+c$ $p+r$
$a+c$ $q-r+s$	b $p+r$	$b-c+d$ $q\ \ +s$	$a\ \ +d$ p	$b-c$ $q-r$	a $p+r+s$	$a+c+d$ q	$b\ \ +d$ $p\ \ +s$
$b-c$ $q\ \ +s$	a p	$a+c+d$ $q-r+s$	$b\ \ +d$ $p+r$	$a+c$ q	b $p\ \ +s$	$b-c+d$ $q-r$	$a\ \ +d$ $p+r+s$

Fig. 0.1: Tarry's patterns from 1903

The eight numbers a, $b-c$, $b+d$, ... have to be chosen among 1, 2, ..., 8 and the numbers $p+r$, $q-r+s$, p, ... among 0, 8, ..., 56. If you choose for example

Solution Set							
a	b	c	d	p	q	r	s
1	4	1	4	0	24	8	32

you will get the pandiagonal bimagic square from figure 0.2.

9	51	8	62	44	18	37	31
4	58	13	55	33	27	48	22
46	24	35	25	15	53	2	60
39	29	42	20	6	64	11	49
21	47	28	34	56	14	57	3
32	38	17	43	61	7	52	10
50	12	63	5	19	41	30	40
59	1	54	16	26	36	23	45

Fig. 0.2: Pandiagonal bimagic square

[1] Tarry [3] S. 141 – 142

Francis Gaspalou[2] analyzed these squares and found 320 pandiagonal bimagic squares, 80 of them being really unique.

One year later, Tarry published ten more pattern schemes[3], where first one is shown is figure 0.3.

$b - c + d$ $cp - r(a-b)$	$a + c$ $cp + cr$	$b + d$ $cp + cs$	a $cp - r(a-b+c) + cs$	$a + c + d$ $cp - r(a-b) + cs$	$b - c$ $cp + cr + cs$	$a + d$ cp	b $cp - r(a-b+c)$
a $cp - r(a-b) + cs$	$b + d$ $cp + cr + cs$	$a + c$ cp	$b - c + d$ $cp - r(a-b+c)$	b $cp - r(a-b)$	$a + d$ $cp + cr$	$b - c$ $cp + cs$	$a + c + d$ $cp - r(a-b+c) + cs$
$a + c + d$ $cp + cr$	$b - c$ $cp - r(a-b)$	$a + d$ $cp - r(a-b+c) + cs$	b $cp + cs$	$b - c + d$ $cp + cr + cs$	$a + c$ $cp - r(a-b) + cs$	$b + d$ $cp - r(a-b+c)$	a cp
b $cp + cr + cs$	$a + d$ $cp - r(a-b) + cs$	$b - c$ $cp - r(a-b+c)$	$a + c + d$ cp	a $cp + cr$	$b + d$ $cp - r(a-b)$	$a + c$ $cp - r(a-b+c) + cs$	$b - c + d$ $cp + cs$
$b - c$ cp	$a + c + d$ $cp - r(a-b+c)$	b $cp - r(a-b) + cs$	$a + d$ $cp + cr + cs$	$a + c$ $cp + cs$	$b - c + d$ $cp - r(a-b+c) + cs$	a $cp - r(a-b)$	$b + d$ $cp + cr$
$a + d$ $cp + cs$	b $cp - r(a-b+c) + cs$	$a + c + d$ $cp - r(a-b)$	$b - c$ $cp + cr$	$b + d$ cp	a $cp - r(a-b+c)$	$b - c + d$ $cp - r(a-b) + cs$	$a + c$ $cp + cr + cs$
$a + c$ $cp - r(a-b+c)$	$b - c + d$ cp	a $cp + cr + cs$	$b + d$ $cp - r(a-b) + cs$	$b - c$ $cp - r(a-b+c) + cs$	$a + c + d$ $cp + cs$	b $cp + cr$	$a + d$ $cp - r(a-b)$
$b + d$ $cp - r(a-b+c) + cs$	a $cp + cs$	$b - c + d$ $cp + cr$	$a + c$ $cp - r(a-b)$	$a + d$ $cp - r(a-b+c)$	b cp	$a + c + d$ $cp + cr + cs$	$b - c$ $cp - r(a-b) + cs$

Fig. 0.3: Scheme #1 of Tarry's patterns from 1904

I analyzed these patterns and found out that each of them will also create 320 pandiagonal bimagic squares, 80 of them being really unique again. For example, the following solution set for scheme #1

Solution Set							
a	b	c	d	p	q	r	s
1	4	1	4	0	-64	8	32

will produce the auxiliary squares

7	2	8	1	6	3	5	4
1	8	2	7	4	5	3	6
6	3	5	4	7	2	8	1
4	5	3	6	1	8	2	7
3	6	4	5	2	7	1	8
5	4	6	3	8	1	7	2
2	7	1	8	3	6	4	5
8	1	7	2	5	4	6	3

24	8	32	48	56	40	0	16
56	40	0	16	24	8	32	48
8	24	48	32	40	56	16	0
40	56	16	0	8	24	48	32
0	16	56	40	32	48	24	8
32	48	24	8	0	16	56	40
16	0	40	56	48	32	8	24
48	32	8	24	16	0	40	56

Fig. 0.4: Auxiliary squares from scheme #1

[2] Gaspalou [2]

[3] Tarry [4]

which add together to the pandiagonal bimagic square in figure 0.5.

31	10	40	49	62	43	5	20
57	48	2	23	28	13	35	54
14	27	53	36	47	58	24	1
44	61	19	6	9	32	50	39
3	22	60	45	34	55	25	16
37	52	30	11	8	17	63	42
18	7	41	64	51	38	12	29
56	33	15	26	21	4	46	59

Fig. 0.5: Pandiagonal bimagic sqare from scheme #1 of Tarry's patterns

Let me give another example from scheme #2, which is shown in figure 0.6.

$a + c$ $cp + cs$	$b - c$ $cp + cr + cs$	$a + d$ cp	$b + d$ $cp + cr$	$a + c + d$ $cp - r(a - b) + cs$	$b - c + d$ $cp - r(a - b + c) + cs$	a $cp - r(a - b)$	b $cp - r(a - b + c)$
b $cp - r(a - b)$	a $cp - r(a - b + c)$	$b - c + d$ $cp - r(a - b) + cs$	$a + c + d$ $cp - r(a - b + c) + cs$	$b + d$ cp	$a + d$ $cp + cr$	$b - c$ $cp + cs$	$a + c$ $cp + cr + cs$
$b - c + d$ $cp + cr + cs$	$a + c + d$ $cp + cs$	b $cp + cr$	a cp	$b - c$ $cp - r(a - b + c) + cs$	$a + c$ $cp - r(a - b) + cs$	$b + d$ $cp - r(a - b + c)$	$a + d$ $cp - r(a - b)$
$a + d$ $cp - r(a - b + c)$	$b + d$ $cp - r(a - b)$	$a + c$ $cp - r(a - b + c) + cs$	$b - c$ $cp - r(a - b) + cs$	a $cp + cr$	b cp	$a + c + d$ $cp + cr + cs$	$b - c + d$ $cp + cs$
$b - c$ cp	$a + c$ $cp + cr$	$b + d$ $cp + cs$	$a + d$ $cp + cr + cs$	$b - c + d$ $cp - r(a - b)$	$a + c + d$ $cp - r(a - b + c)$	b $cp - r(a - b) + cs$	a $cp - r(a - b + c) + cs$
a $cp - r(a - b) + cs$	b $cp - r(a - b + c) + cs$	$a + c + d$ $cp - r(a - b)$	$b - c + d$ $cp - r(a - b + c)$	$a + d$ $cp + cs$	$b + d$ $cp + cr + cs$	$a + c$ cp	$b - c$ $cp + cr$
$a + c + d$ $cp + cr$	$b - c + d$ cp	a $cp + cr + cs$	b $cp + cs$	$a + c$ $cp - r(a - b + c)$	$b - c$ $cp - r(a - b)$	$a + d$ $cp - r(a - b + c) + cs$	$b + d$ $cp - r(a - b) + cs$
$b + d$ $cp - r(a - b + c) + cs$	$a + d$ $cp - r(a - b) + cs$	$b - c$ $cp - r(a - b + c)$	$a + c$ $cp - r(a - b)$	b $cp + cr + cs$	a $cp + cs$	$b - c + d$ $cp + cr$	$a + c + d$ cp

Fig. 0.6: Scheme #2 of Tarry's patterns from 1904

The solution set

Solution Set							
a	b	c	d	p	q	r	s
3	5	-2	1	-20	-64	4	16

will produce the auxiliary squares

1	7	4	6	2	8	3	5
5	3	8	2	6	4	7	1
8	2	5	3	7	1	6	4
4	6	1	7	3	5	2	8
7	1	6	4	8	2	5	3
3	5	2	8	4	6	1	7
2	8	3	5	1	7	4	6
6	4	7	1	5	3	8	2

8	0	40	32	16	24	48	56
48	56	16	24	40	32	8	0
0	8	32	40	24	16	56	48
56	48	24	16	32	40	0	8
40	32	8	0	48	56	16	24
16	24	48	56	8	0	40	32
32	40	0	8	56	48	24	16
24	16	56	48	0	8	32	40

Fig. 0.7: Auxiliary squares from scheme #2

which add together to the pandiagonal bimagic square in figure 0.8.

9	7	44	38	18	32	51	61
53	59	24	26	46	36	15	1
8	10	37	43	31	17	62	52
60	54	25	23	35	45	2	16
47	33	14	4	56	58	21	27
19	29	50	64	12	6	41	39
34	48	3	13	57	55	28	22
30	20	63	49	5	11	40	42

Fig. 0.8: Pandiagonal bimagic square from scheme #2 of Tarry's patterns

Tarry's ten different schemes will produce $320 \cdot 10 = 3200$ bimagic squares altogether, but only 320 of them are really unique.

Those 320 unique squares can also be derived from the pattern schemes #1, #2, #5 and #6. Furthermore, I found that the squares from #2 and #3 are the same as those from Tarry's patterns in [3].

In 1934 Cazalas translated these patterns into numbers[4] and with his modified version of Tarry's number series, he was able to produce some of these squares, but his results couldn't reach the great variety of Tarry's patterns.

The basic values of the number series corresponding to the ten schemes of Tarry are shown in figure 0.9. For a deeper understanding of the mentioned number series please look into the book of Cazalas.

[4] Cazalas [1] S. 109 – 111

N^{os}	r_1	r_2	r_3	s_1	s_2	s_3		$+K$
1	010111	111001	100011	100110	010011	011100		011110
2	111010	001111	100011	001011	100101	011100		010011
3	001101	010011	101010	111100	001111	010101		010001
4	010110	100101	101010	100111	111001	010101		100101
5	110001	011100	001011	100111	111001	110100	(base 2)	010011
6	101100	110010	001011	111010	010111	110100		111001
7	001110	100011	011001	111100	001111	100110		011010
8	100101	010110	011001	010111	111010	100110		100101
9	111100	010111	110001	101001	110100	001110		101001
10	100111	111010	110001	110010	011001	001110		111010

Fig. 0.9: Basic values for the modified number series of Cazalas

References

[1] CAZALAS, Général E. *Carrés Magiques au degre n*. Paris: Hermann et Cie, 1934.

[2] GASPALOU, Francis. *How Many Squares Are There, Mr. Tarry?* 2012. URL: `http://www.multimagie.com/GaspalouTarry.pdf`.

[3] TARRY, Gaston. *Carrés bimagiques de base 3n*. In: *Association Française pour l'Avancement des Sciences*. Paris, 1903, pp. 130–142.

[4] TARRY, Gaston. *Carrés cabalistique eulérien de base 8n*. In: *Association Française pour l'Avancement des Sciences*. Paris, 1904, pp. 95–111.

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