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On squares of squares II

by

Andrew Bremner (Tempe, AZ)

0. The problem of determining whether there can exist a 3×3 magic square whose entries are perfect squares is an intriguing one, and has been discussed by several authors, including Bremner [3], Gardner [6], Guy & Nowakowski [7], LaBar [9], Robertson [10], Sallows [11]. There are two underlying problems. First, to find 3×3 squares all of whose entries are square, and with as many row, column, and diagonal sums as possible being equal. Second, to find 3×3 magic squares with as many entries as possible being perfect squares. The first problem was discussed in some detail in Bremner [3], and it is the intention of this current writing to address the second problem. A 3×3 square is said to be *trivial* if it contains repeated entries; there apparently is known just one non-trivial magic square (together with its symmetries) with seven square entries:

$$\begin{bmatrix} 373^2 & 289^2 & 565^2 \\ 360721 & 425^2 & 23^2 \\ 205^2 & 527^2 & 222121 \end{bmatrix} \tag{1}$$

and no examples known of non-trivial squares with eight square entries, unless one extends the ground field when there are examples such as:

$$\begin{bmatrix} (22+4\sqrt{3}\,)^2 & (17-9\sqrt{3}\,)^2 & (5+13\sqrt{3}\,)^2 \\ (23-\sqrt{3}\,)^2 & 2^2\cdot 7\cdot 19 & (23+\sqrt{3}\,)^2 \\ (5-13\sqrt{3}\,)^2 & (17+9\sqrt{3}\,)^2 & (22-4\sqrt{3}\,)^2 \end{bmatrix} \tag{2}$$

over the field $\mathbb{Q}(\sqrt{3}\,)$.

Demanding that *six* given entries be square turns out to be equivalent to studying the intersection of three quadrics in five-dimensional projective space. By studying this surface in each of the sixteen ways (up to symmetry) of selecting six entries from a 3×3 square, we show that for each of these sixteen configurations there are infinitely many magic squares with the

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six entries being perfect squares. This is done by parametrizations in one variable, so that asking for a seventh entry to be square in these examples involves finding rational points on hyperelliptic curves (in general of high genus) of type $f(t) = \square$. But we have been unable to find any magic squares with seven square entries, other than the example at (1).

We further investigate in detail the geometry underlying one of the sixteen configurations; and this allows construction of an infinite family of eight-square magic squares over $\mathbb{Q}(\sqrt{3})$, such as (2).

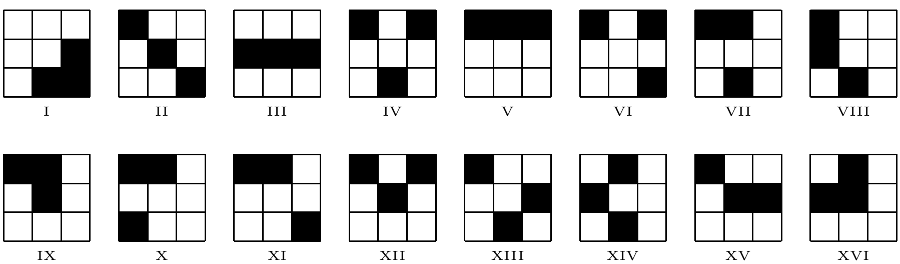
1. Any three-by-three magic square of rational numbers has the form

$$\begin{bmatrix} a+c & -a-b+c & b+c \\ -a+b+c & c & a-b+c \\ -b+c & a+b+c & -a+c \end{bmatrix} \tag{3}$$

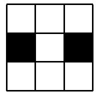
with $a, b, c \in \mathbb{Q}$. (In all that follows, a "magic square" will refer to a square with rational entries, unless specifically otherwise indicated.) The square (3) is trivial (has repeated entries) precisely when

$$ab(a^2-b^2)(a^2-4b^2)(4a^2-b^2) = 0.$$

Up to symmetry (rotation and reflection), there are precisely sixteen ways of selecting six entries from a 3×3 square. These are as follows:



For each of these sixteen configurations, it is possible to find magic squares with the six selected entries perfect squares: see Figure 1 (where we list examples which are believed in each case to have minimum "magic" sum). We shall show that it is possible to find infinitely many parametrized examples of such squares in each of the sixteen cases I, ..., XVI. Originally, it was hoped that squares with seven square entries might be found by extending a square with six square entries; but of all the examples considered, and after considerable computing, only the square at (1) arose. (It is worth noting that Martin Gardner [6] has offered \$100 for an example of a non-trivial nine square entry magic square, or a proof of non-existence of such.) Duncan Buell [5] has shown by careful search that there is no seven-square magic square corresponding to the "hour-glass" configuration



in which the central element of the square is less than $25 \cdot 10^{24}$.

The demand that each entry at (3) be a perfect square results in nine equations, which, on eliminating a, b, c, is equivalent to the intersection of six quadrics in eight-dimensional projective space, a *surface*. It seems difficult to deduce properties of this surface of high degree. Recall the notorious problem of finding a cubical box with all edges, face diagonals and box diagonal rational: the corresponding equations are four quadrics in six-dimensional projective space. Here, we restrict to considering just six entries from (3) being squares. On eliminating a, b, c there now results the intersection of three quadrics in $\mathbb{P}^5$, and there is some hope of applying to this surface elementary ideas from geometry. In particular, if the surface is non-singular, then it is $K3$ and the methods of Swinnerton-Dyer [16] may be of aid.

2. The arithmetic techniques used to analyze each of the sixteen configurations are similar, and so detailed attention will be restricted to just a few cases.

Consider the configuration II, in which

$$\pm b + c = \square, \qquad \pm a \pm b + c = \square.$$

Put

$$a \pm b + c = (g \pm h)^2, \qquad -a \pm b + c = (r \pm s)^2, \qquad \pm b + c = (m \pm n)^2,$$

so that

$$a = \tfrac{1}{2}(g^2 + h^2 - r^2 - s^2), \qquad b/2 = gh = mn = rs,$$
$$c = m^2 + n^2 = \tfrac{1}{2}(g^2 + h^2 + r^2 + s^2).$$

From $g/m = n/h = \beta/\gamma$, say, follows $g = \alpha\beta, m = \alpha\gamma, n = \beta\delta, h = \gamma\delta$ with

$$a = (\alpha^2 - \delta^2)(\beta^2 - \gamma^2), \qquad b = 2\alpha\beta\gamma\delta, \qquad c = \alpha^2\gamma^2 + \beta^2\delta^2$$

and

$$\begin{aligned} &\alpha^2(-\beta^2 + 2\gamma^2) + 2\alpha\beta\gamma\delta + \delta^2(2\beta^2 - \gamma^2) = (r+s)^2, \\ &\alpha^2(-\beta^2 + 2\gamma^2) - 2\alpha\beta\gamma\delta + \delta^2(2\beta^2 - \gamma^2) = (r-s)^2, \end{aligned} \tag{4}$$

that is,

$$E: \quad \begin{cases} \alpha^2(2 - \lambda^2) + 2\lambda\alpha\delta - \delta^2(1 - 2\lambda^2) = (\varrho + \sigma)^2, \\ \alpha^2(2 - \lambda^2) - 2\lambda\alpha\delta - \delta^2(1 - 2\lambda^2) = (\varrho - \sigma)^2, \end{cases} \tag{5}$$

where we have put $\lambda = \beta/\gamma$, $\varrho = r/\gamma$, $\sigma = s/\gamma$. Regarded as the intersection of two quadrics in $\alpha,\ \delta,\ \varrho,\ \sigma$-space over $\mathbb{Q}(\lambda)$, E represents an elliptic curve

I

49^2	143^2	155^2
193^2	125^2	-10999
85^2	10801	28849

V

1153	1057	313
1^2	29^2	41^2
37^2	25^2	23^2

II

1945	1^2	37^2
23^2	1105	41^2
29^2	47^2	265

VI

3385	47^2	7081
89^2	65^2	23^2
37^2	79^2	5065

III

541^2	421^2	49^2
-132839	157441	447721
559^2	371^2	149^2

VII

889	697	17^2
5^2	25^2	35^2
31^2	553	19^2

IV

93961	191^2	43801
89^2	241^2	329^2
269^2	79681	149^2

VIII

1561	31^2	1^2
-719	29^2	49^2
41^2	721	11^2

Fig. 1. The sixteen configurations of six square entries in a magic square

IX

2713	673	35^2
7^2	1537	55^2
43^2	49^2	19^2

XIII

313	23^2	41^2
47^2	29^2	-527
1^2	1153	37^2

X

3001	-1679	61^2
49^2	41^2	31^2
-359	71^2	19^2

XIV

5^2	1561	17^2
889	25^2	19^2
31^2	-311	35^2

XI

10585	-1679	113^2
97^2	85^2	71^2
41^2	127^2	3865

XV

265	1^2	13^2
7^2	145	241
11^2	17^2	5^2

XII

22009	119^2	9265
49^2	15145	167^2
145^2	127^2	91^2

XVI

37^2	5089	67^2
6769	3649	23^2
53^2	47^2	77^2

Fig. 1 [cont.]

with distinguished point $(\alpha, \delta, \varrho, \sigma) = (1, 1, \lambda, 1)$. A cubic equation for the curve is given by

$$E: \quad T^2 = S(S^2 - 4(1 - 3\lambda^2 + \lambda^4)S + 4(1 - \lambda^2)^4),$$

and it is straightforward to verify that $(2(1-\lambda^2)^2, 4\lambda(1-\lambda^2)^2)$ is of order 4 in $G = E(\mathbb{Q}(\lambda))$, with the torsion subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z}$; and the rank of E over $\mathbb{Q}(\lambda)$ is equal to 1. The point $(2(1+\lambda)^4, 12\lambda(1+\lambda)^4)$ is of infinite order in G, and likely a generator of the infinite component of G, though this has not been confirmed. On (5) this point corresponds to $P = (1, 1, 1, \lambda)$. The multiples of P on (5) give rise to points $(\alpha, \delta, \varrho, \sigma)$ which in turn give rise to a, b, c and corresponding magic squares for the configuration II. For example, $-P$ is equal to

$$((-5+\lambda^2)(1+7\lambda^2), (1-5\lambda^2)(7+\lambda^2), (1-5\lambda^2)(1+7\lambda^2), \lambda(-5+\lambda^2)(7+\lambda^2))$$

with

$$\begin{aligned}
a &= 24(1 - \lambda^2)^2(1 + \lambda^2)(1 - 6\lambda + \lambda^2)(1 + 6\lambda + \lambda^2),\\
b &= -2\lambda(5 - \lambda^2)(1 - 5\lambda^2)(1 + 7\lambda^2)(7 + \lambda^2),\\
c &= (1 + \lambda^2)(5 - 6\lambda + 5\lambda^2)(5 + 6\lambda + 5\lambda^2)(1 + 14\lambda^2 + \lambda^4);
\end{aligned}$$

and the corresponding square (m_{ij}) is

(6)

$$\begin{bmatrix}
* & (1+35\lambda+2\lambda^2-2\lambda^3-35\lambda^4-\lambda^5)^2 & (5-7\lambda+34\lambda^2+34\lambda^3-7\lambda^4+5\lambda^5)^2\\
(1-35\lambda+2\lambda^2+2\lambda^3-35\lambda^4+\lambda^5)^2 & * & (7+5\lambda-34\lambda^2+34\lambda^3-5\lambda^4-7\lambda^5)^2\\
(5+7\lambda+34\lambda^2-34\lambda^3-7\lambda^4-5\lambda^5)^2 & (7-5\lambda-34\lambda^2-34\lambda^3-5\lambda^4+7\lambda^5)^2 & *
\end{bmatrix}$$

where the diagonal entries are

$$\begin{aligned}
m_{11} &= 49 - 451\lambda^2 + 1426\lambda^4 + 1426\lambda^6 - 451\lambda^8 + 49\lambda^{10},\\
m_{22} &= 25 + 389\lambda^2 + 610\lambda^4 + 610\lambda^6 + 389\lambda^8 + 25\lambda^{10},\\
m_{33} &= 1 + 1229\lambda^2 - 206\lambda^4 - 206\lambda^6 + 1229\lambda^8 + \lambda^{10}.
\end{aligned} \tag{7}$$

The magic squares that arise in this way from points Q and $Q+T$ in G for T torsion, are symmetries of each other; to compute all magic squares arising from E it is only necessary therefore to consider points $Q = mP$. One can even restrict to $m > 0$; P produces a trivial square, $2P$ produces (up to symmetry) the square at (6) with entries of degree 10, and $3P$ produces a square with entries of degree 26.

It is possible to achieve a square with entries of degree 8 by means of the following specialization. Put

$$(\beta, \gamma) = (1 + 2\mu + 3\mu^2, 2\mu).$$

Then (4) takes the form

$$(8)\quad \begin{aligned}&\alpha^2(-1-4\mu-2\mu^2-12\mu^3-9\mu^4)+4\mu(1+2\mu+3\mu^2)\alpha\delta\\ &\qquad\qquad +2\delta^2(1+4\mu+8\mu^2+12\mu^3+9\mu^4)=(r+s)^2,\\ &\alpha^2(-1-4\mu-2\mu^2-12\mu^3-9\mu^4)-4\mu(1+2\mu+3\mu^2)\alpha\delta\\ &\qquad\qquad +2\delta^2(1+4\mu+8\mu^2+12\mu^3+9\mu^4)=(r-s)^2\end{aligned}$$

with point at $(\alpha,\delta)=(-1+3\mu^2,1-2\mu+3\mu^2)$. This leads to the magic square

$$(9)\quad \begin{bmatrix} * & (1+4\mu^2+12\mu^3-9\mu^4)^2 & (1-2\mu+2\mu^2+6\mu^3+9\mu^4)^2\\ (1-4\mu-4\mu^2-9\mu^4)^2 & * & (1+4\mu-4\mu^2-9\mu^4)^2\\ (1+2\mu+2\mu^2-6\mu^3+9\mu^4)^2 & (1+4\mu^2-12\mu^3-9\mu^4)^2 & * \end{bmatrix}$$

where the diagonal entries are

$$(10)\quad \begin{aligned} m_{11}&=1+4\mu+8\mu^2-28\mu^3-2\mu^4-84\mu^5+72\mu^6+108\mu^7+81\mu^8,\\ m_{22}&=(1+\mu^2)(1+9\mu^2)(1-2\mu^2+9\mu^4),\\ m_{33}&=1-4\mu+8\mu^2+28\mu^3-2\mu^4+84\mu^5+72\mu^6-108\mu^7+81\mu^8.\end{aligned}$$

The curve at (8) has rank 2 over $\mathbb{Q}(\mu)$ with independent points of infinite order given by

$$\begin{aligned}&(1,1,1+4\mu+3\mu^2,-1-3\mu^2),\\ &(-1+3\mu^2,1-2\mu+3\mu^2,1-4\mu-4\mu^2-9\mu^4,1+4\mu^2+12\mu^3+9\mu^4)\end{aligned}$$

(where we have chosen $(1,1,1+4\mu+3\mu^2,1+3\mu^2)$ as the zero of the group of points over $\mathbb{Q}(\mu)$). Correspondingly, there arises a two-dimensional family of magic squares, with those of smaller degree having entries of degree $8,10,12,16,20,\dots$

Remark. From (10), the square at (9) has m_{22} a perfect square provided the curve $(1+X)(1+9X)(1-2X+9X^2)=\square$ has rational points with $X=\mu^2$. But this elliptic curve of conductor 48 has rank 0, forcing $\mu=0$ and a consequent trivial square. For m_{11} or m_{33} to be square, the condition is that the curve of genus 3 given by $1+4x+8x^2-28x^3-2x^4-84x^5+72x^6+108x^7+81x^8=\square$ have rational points. The curve does of course have only finitely many rational points; it seems likely that $x=0,\infty$ are the only such, though we are unable to show this. The curve contains in its Jacobian the elliptic curve $U^4+4U^3-4U^2-64U-32=\square$ (observe the transformation $U=3x+1/x$), of conductor 1104 and rank 1, with $(-2,8)$ a generator for the rational points. But computing the "small" rational points on this curve led to no non-trivial rational points on the curve of genus 3.

Each of the sixteen configurations I–XVI may be treated as in the above example for Category II. In each case it is straightforward to determine an elliptic fibration of the associated surface intersection of the three quadratic forms. To shorten the labour, observe that there is a correspondence between squares of type II and squares of type XIII, namely

$$\begin{array}{|c|c|c|}\hline m_{11} & B & C \\ \hline D & m_{22} & F \\ \hline G & H & m_{33} \\ \hline \end{array} \quad \longleftrightarrow \quad \begin{array}{|c|c|c|}\hline n_{11} & D & F \\ \hline H & G & n_{23} \\ \hline B & n_{32} & C \\ \hline \end{array}$$

with

$$\begin{aligned} m_{11} &= (F+H)/2, & n_{11} &= B-D+G, \\ m_{22} &= (C+G)/2, & n_{23} &= B+F-H, \\ m_{33} &= (B+D)/2, & n_{32} &= F+G-C. \end{aligned}$$

Similarly, there is a correspondence between squares of type VII and XIV:

$$\begin{array}{|c|c|c|}\hline m_{11} & m_{12} & C \\ \hline D & E & F \\ \hline G & m_{32} & I \\ \hline \end{array} \quad \longleftrightarrow \quad \begin{array}{|c|c|c|}\hline D & n_{12} & C \\ \hline n_{21} & E & I \\ \hline G & n_{32} & F \\ \hline \end{array}$$

with

$$\begin{aligned} m_{11} &= E+F-G, & n_{12} &= -D+F+I, \\ m_{12} &= -C+D+G, & n_{21} &= C-E+F, \\ m_{32} &= C+F-G, & n_{32} &= C-G+I. \end{aligned}$$

It can also be noted that squares in Category VII occur in pairs:

$$\begin{array}{|c|c|c|}\hline \cdot & \cdot & C \\ \hline D & E & F \\ \hline G & \cdot & I \\ \hline \end{array} \quad \longleftrightarrow \quad \begin{array}{|c|c|c|}\hline \cdot & \cdot & F \\ \hline G & E & C \\ \hline D & \cdot & I \\ \hline \end{array}$$

as do squares in Category XIII:

$$\begin{array}{|c|c|c|}\hline \cdot & B & C \\ \hline D & E & \cdot \\ \hline G & \cdot & I \\ \hline \end{array} \quad \longleftrightarrow \quad \begin{array}{|c|c|c|}\hline \cdot & G & D \\ \hline C & I & \cdot \\ \hline B & \cdot & E \\ \hline \end{array}\,.$$

We find in this manner parametrizations for the squares in Categories I–XVI of respective degrees 12, 8, 12, 20, 12, 12, 12, 20, 20, 12, 20, 20, 8, 12, 20, 12 (but do not claim in any particular Category to have found the parametrization of smallest degree). For reasons of space, these parametrizations are not listed here, but are available to interested parties from the author.

Finally we consider one further example, squares of type VII. Here

$$\pm b + c = \square, \qquad \pm(a-b) + c = \square, \qquad -a + c = \square, \qquad c = \square$$

and putting

$$c = m^2 + n^2 = r^2 + s^2 = u^2, \qquad b = 2mn, \qquad a - b = 2rs, \qquad a = u^2 - v^2,$$

we have

$$m^2 + n^2 = r^2 + s^2 = u^2, \quad u^2 - v^2 = 2mn + 2rs.$$

With

$$m = \alpha\beta + \gamma\delta, \qquad n = \alpha\gamma - \beta\delta, \qquad r = \alpha\gamma + \beta\delta, \qquad s = \alpha\beta - \gamma\delta$$

we obtain

$$\begin{aligned} (\alpha^2 + \delta^2)(\beta^2 + \gamma^2) &= u^2, \\ \alpha^2(\beta^2 - 4\beta\gamma + \gamma^2) + \delta^2(\beta^2 + 4\beta\gamma + \gamma^2) &= v^2, \end{aligned}$$

that is,

$$\text{(11)} \qquad F: \quad \begin{cases} (\alpha^2 + \delta^2)(1 + \lambda^2) = \varrho^2, \\ \alpha^2(1 - 4\lambda + \lambda^2) + \delta^2(1 + 4\lambda + \lambda^2) = \sigma^2, \end{cases}$$

where we have put $\lambda = \beta/\gamma, \varrho = u/\gamma, \sigma = v/\gamma$. As a curve in $\alpha, \delta, \varrho, \sigma$-space over $\mathbb{Q}(\lambda)$, (11) is an elliptic curve with distinguished point $(\alpha, \delta, \varrho, \sigma) = (\lambda, 1, 1 + \lambda^2, 1 + 2\lambda - \lambda^2)$. The $\mathbb{Q}(\lambda)$-rank turns out to equal 2, with independent points of infinite order $R = (1, \lambda, 1 + \lambda^2, 1 - 2\lambda - \lambda^2), S = (1, -\lambda, 1 + \lambda^2, 1 - 2\lambda - \lambda^2)$. Then $-S$ has

$$(\alpha, \delta) = (1 + 4\lambda + 2\lambda^2 + 4\lambda^3 - 3\lambda^4, \lambda(3 + 4\lambda - 2\lambda^2 + 4\lambda^3 - \lambda^4))$$

leading to the square

$$\begin{bmatrix} * & * & (1+\lambda^2)^2(1+8\lambda+6\lambda^2-8\lambda^3+\lambda^4)^2 \\ (1+6\lambda+5\lambda^2+4\lambda^3-5\lambda^4+6\lambda^5-\lambda^6)^2 & (1+\lambda^2)^2(1+4\lambda+6\lambda^2-4\lambda^3+\lambda^4)^2 & (1+2\lambda-\lambda^2)^2(1+6\lambda^2+\lambda^4)^2 \\ (1+\lambda^2)^2(1-10\lambda^2+\lambda^4)^2 & * & (1+2\lambda-3\lambda^2+12\lambda^3+3\lambda^4+2\lambda^5-\lambda^6)^2 \end{bmatrix}$$

where the non-square entries are

$$\begin{aligned} m_{11} &= (1 + 2\lambda - \lambda^2)(1 + 10\lambda + 43\lambda^2 + 24\lambda^3 + 58\lambda^4 + 60\lambda^5 - 58\lambda^6 \\ &\quad + 24\lambda^7 - 43\lambda^8 + 10\lambda^9 - \lambda^{10}), \\ m_{12} &= (1 - 10\lambda + 5\lambda^2 - 12\lambda^3 - 5\lambda^4 + 6\lambda^5 - \lambda^6) \\ &\quad \times (1 + 6\lambda + 5\lambda^2 - 12\lambda^3 - 5\lambda^4 - 10\lambda^5 - \lambda^6), \\ m_{32} &= (1 + 2\lambda - \lambda^2)(1 + 18\lambda + 75\lambda^2 + 24\lambda^3 + 90\lambda^4 + 44\lambda^5 - 90\lambda^6 \\ &\quad + 24\lambda^7 - 75\lambda^8 + 18\lambda^9 - \lambda^{10}). \end{aligned}$$

Computing "small" combinations of R, S on F leads to squares with entries of degree 12, 20, 28, ...

The first equation at (11) may be parametrized by

$$\alpha : \delta : \varrho = \lambda(p^2 - q^2) + 2pq : (p^2 - q^2) - 2\lambda pq : (\lambda^2 + 1)(p^2 + q^2)$$

and then the second equation at (11) demands

$$(12) \quad (1+2\lambda-\lambda^2)^2 p^4 - 32\lambda^2 p^3 q + 2(1-12\lambda+2\lambda^2+12\lambda^3+\lambda^4)p^2q^2 + 32\lambda^2 pq^3 + (1+2\lambda-\lambda^2)^2 q^4 = \Box.$$

The conditions that the remaining three entries in this Category VII square be perfect squares have become:

$$(13) \quad \begin{aligned} &(1-2\lambda-\lambda^2)^2 p^4 + 32\lambda^2 p^3 q + 2(1+12\lambda+2\lambda^2-12\lambda^3+\lambda^4)p^2q^2 \\ &\qquad - 32\lambda^2 pq^3 + (1-2\lambda-\lambda^2)^2 q^4 = \Box, \\ &(1+2\lambda-\lambda^2)^2 p^4 - 4(1+10\lambda^2+\lambda^4)p^3 q + 2(1-12\lambda+2\lambda^2+12\lambda^3+\lambda^4)p^2q^2 \\ &\qquad + 4(1+10\lambda^2+\lambda^4)pq^3 + (1+2\lambda-\lambda^2)^2 q^4 = \Box, \\ &(1-2\lambda-\lambda^2)^2 p^4 + 4(1+10\lambda^2+\lambda^4)p^3 q + 2(1+12\lambda+2\lambda^2-12\lambda^3+\lambda^4)p^2q^2 \\ &\qquad - 4(1+10\lambda^2+\lambda^4)pq^3 + (1-2\lambda-\lambda^2)^2 q^4 = \Box. \end{aligned}$$

In order to try and find a seven-square magic square, (12) was searched for solutions also satisfying at least one of the equations (13). The only solution (up to symmetry) occurred at $\lambda = 13$ with $(p, q) = (9, 2)$, giving rise to the square at (1). (By symmetry considerations, it is sufficient to search over a region in which p, q, and $n(\lambda) = \text{numerator}(\lambda)$, $d(\lambda) = \text{denominator}(\lambda)$ are positive; the search covered the region $p + q + n(\lambda) + d(\lambda) \le 1000$.)

3. We use geometric techniques to analyze squares in Category III. Here,

$$\pm a + c = \Box, \qquad \pm b + c = \Box, \qquad \pm(a+b) + c = \Box,$$

and putting

$$a = 2TU, \quad b = 2VW, \quad a+b = -2XY, \quad c = T^2+U^2 = V^2+W^2 = X^2+Y^2$$

there results

$$\mathcal{S}: \quad T^2 + U^2 = V^2 + W^2 = X^2 + Y^2, \qquad TU + VW + XY = 0.$$

This intersection of three quadrics in $\mathbb{P}^5$ is the equation of a surface. It is readily determined to be non-singular, and so $\mathcal{S}$ is $K3$. There is a large symmetry group on $\mathcal{S}$, with the obvious symmetries $(TU) \leftrightarrow (UT)$, etc., $(TUVW) \leftrightarrow (VWTU)$, etc., $(TUVWXY) \leftrightarrow (VWXYTU)$, etc., and sign-changes $(TVX) \leftrightarrow (-T-V-X)$, etc., generating a group of order 384. There are 24 conics on $\mathcal{S}$, four lying in each of the six hyperplanes $T = 0$, $U = 0, \ldots, Y = 0$, and typified by $\{T = X, U = -Y, V = 0, W^2 = X^2+Y^2\}$, which will be denoted by C_{TXV}, where C_{rst} denotes the conic with $r = s$, $t = 0$. $\mathcal{S}$ also contains the 8 complex conics $\{T = \varepsilon_1 U, V = \varepsilon_2 W, X = \varepsilon_3 Y, \varepsilon_1 U^2 + \varepsilon_2 W^2 + \varepsilon_3 Y^2 = 0\}$ where ε_i are square roots of -1. We denote these conics by $C_{j_1 j_2 j_3}$ with $j_k = +, -$ according as $\varepsilon_k = i, -i$.

There are also 32 straight lines on $\mathcal{S}$, typified up to symmetry by the following equations:

$$L_1: \quad \{T+X=V, T-X=-\sqrt{3}\,W, U+Y=W, U-Y=\sqrt{3}\,V\}$$

with parametrization

$$(T,U,V,W,X,Y)=(\sqrt{3}\,\alpha, \alpha+2\beta, \sqrt{3}\,(\alpha+\beta), -\alpha+\beta, \sqrt{3}\,\beta, -2\alpha-\beta). \tag{14}$$

Consider now the intersection of $\mathcal{S}$ with the hyperplane

$$U-Y=\lambda(X+T). \tag{15}$$

This cuts out on $\mathcal{S}$ the quadrics C_{UYV} and C_{UYW}, together with a residual intersection of degree 4, on which

$$U+Y=\frac{1}{\lambda}(X-T),$$

and

$$\mathcal{E}_\lambda: \quad \begin{cases} (1+2\lambda-\lambda^2)^2T^2-2(1-\lambda^4)TX+(1-2\lambda-\lambda^2)^2X^2=4\lambda^2(V+W)^2, \\ (1-2\lambda-\lambda^2)^2T^2-2(1-\lambda^4)TX+(1+2\lambda-\lambda^2)^2X^2=4\lambda^2(V-W)^2. \end{cases}$$

As the intersection of two quadrics in $\mathbb{P}^3$ with distinguished point

$$\mathcal{O}_\lambda(T,X,V,W)=(2\lambda,0,1-\lambda^2,2\lambda),$$

$\mathcal{E}_\lambda$ represents the equation of an elliptic curve over $\mathbb{C}(\lambda)$. The curve is singular precisely at $\lambda=\pm1,\pm i,\pm\sqrt{3},\pm1/\sqrt{3}$.

At $\lambda=1$, $\mathcal{E}_\lambda$ decomposes as the sum

$$\mathcal{E}_1: \quad C_{UXV}+C_{UXW}. \tag{16}$$

Similarly,

$$\mathcal{E}_{-1}: \quad C_{TYV}+C_{TYW}. \tag{17}$$

When $\lambda=i$, then $\mathcal{E}_\lambda$ splits as the two conics:

$$\mathcal{E}_i: \quad C_{--+}+C_{-++} \tag{18}$$

with conjugate decomposition at $\lambda=-i$,

$$\mathcal{E}_{-i}: \quad C_{++-}+C_{+--}. \tag{19}$$

When $\lambda=\sqrt{3}$, then $\mathcal{E}_\lambda$ splits as the four lines:

$$\begin{aligned} \mathcal{E}_{\sqrt{3}}: \quad & \{T+X=V, && T-X=-\sqrt{3}\,W, && U+Y=W, && U-Y=\sqrt{3}\,V\} \\ +\,& \{T+X=W, && T-X=-\sqrt{3}\,V, && U+Y=V, && U-Y=\sqrt{3}\,W\} \\ +\,& \{T+X=-W, && T-X=\sqrt{3}\,V, && U+Y=-V, && U-Y=-\sqrt{3}\,W\} \\ +\,& \{T+X=-V, && T-X=\sqrt{3}\,W, && U+Y=-W, && U-Y=-\sqrt{3}\,V\} \end{aligned} \tag{20}$$

with conjugate decomposition at $\lambda=-\sqrt{3}$; and at $\lambda=1/\sqrt{3}$, the decomposition is again four lines corresponding to the symmetry of $\mathcal{S}$ obtained by changing the signs of U,W,X.

A cubic model for $\mathcal{E}_\lambda$ is given by

$$\tau^2 = \sigma(\sigma + (3-\lambda^2)^2)(\sigma + (1-3\lambda^2)^2). \tag{21}$$

The torsion group of $\mathcal{E}_\lambda$ over $\mathbb{C}(\lambda)$ is isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ generated by $T_4(\sigma,\tau) = (-(3-\lambda^2)(1-3\lambda^2),\ -2(1+\lambda^2)(3-\lambda^2)(1-3\lambda^2))$ of order 4, and $T_2(\sigma,\tau) = (-(3-\lambda^2)^2, 0)$ of order 2. These correspond to

$$\begin{aligned} T_4(T,X,V,W) &= (1-\lambda^2, 1+\lambda^2, 1-\lambda^2, -2\lambda),\\ T_2(T,X,V,W) &= (1+\lambda^2, 1-\lambda^2, 2\lambda, 1-\lambda^2). \end{aligned} \tag{22}$$

The four singular fibres of $\mathcal{E}_\lambda$ at $\lambda = \pm 1, \pm i$ are each of Kodaira Type I_2, and of Type I_4 at $\lambda = \pm\sqrt{3}, \pm 1/\sqrt{3}$. Denote the Néron–Severi group of the surface $\mathcal{S}$ over $\mathbb{C}$ by $\mathrm{NS}(\mathcal{S},\mathbb{C})$. Then $\mathrm{NS}(\mathcal{S},\mathbb{C})$ is a finitely generated $\mathbb{Z}$-module, and it follows from Shioda [13] that

$$\operatorname{rank} \mathrm{NS}(\mathcal{S},\mathbb{C}) = \operatorname{rank} \mathcal{E}_\lambda(\mathbb{C}(\lambda)) + 2 + 4\cdot(2-1) + 4\cdot(4-1).$$

Since the rank of the Néron–Severi group of a $K3$-surface cannot exceed 20 (see for example Barth *et al.* [1]), we have $\operatorname{rank} \mathcal{E}_\lambda(\mathbb{C}(\lambda)) \le 2$. Two independent points of infinite order are readily found on $\mathcal{E}_\lambda$, namely

$$\begin{aligned} &P_1(T,X,V,W) = (1-\lambda^2, 1+\lambda^2, 2\lambda, -1+\lambda^2),\\ &P_2(T,X,V,W) = (2-\sqrt{3}+\lambda, 2-\sqrt{3}-\lambda, (1-\sqrt{3}\,)(1+\lambda), (1-\sqrt{3}\,)(1-\lambda)) \end{aligned}$$

and thus $\operatorname{rank} \mathcal{E}_\lambda(\mathbb{C}(\lambda)) = 2$.

Lemma. $\mathcal{E}_\lambda(\mathbb{C}(\lambda)) \simeq \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, *with respective generators* P_1, P_2, T_4, T_2.

Proof. P_1, P_2 correspond respectively to the points

$$\begin{aligned} P_1(\sigma,\tau) &= (-(1+\lambda^2)^2,\ -8\lambda(1-\lambda^4)),\\ P_2(\sigma,\tau) &= ((2+\sqrt{3}\,)(1-\sqrt{3}\,\lambda)(\sqrt{3}+\lambda)(2-\sqrt{3}-\lambda)^2,\\ &\qquad -2(1+\sqrt{3}\,)(1-\lambda)(1-\sqrt{3}\,\lambda)(\sqrt{3}+\lambda)(2-\sqrt{3}-\lambda)(1+\lambda^2)) \end{aligned}$$

on (21), and it is now straightforward to compute canonical heights by the formulae of Silverman [14]. There results $\widehat{h}(P_1) = 1$ and $\widehat{h}(P_2) = 1/2$, $\widehat{h}(P_1+P_2) = 1/2$. The canonical height pairing is defined by

$$\langle P, Q\rangle = \tfrac{1}{2}(\widehat{h}(P+Q) - \widehat{h}(P) - \widehat{h}(Q))$$

and so the height pairing matrix $(\langle P_i, P_j\rangle)_{\{1\le i\le 2,\ 1\le j\le 2\}}$ is equal to

$$\mathcal{H} = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

with $\det(\mathcal{H}) = 1/4$. Thus P_1, P_2 are independent, and by a theorem of Kuwata [8], P_1, P_2 generate a subgroup of index at most a power of 2 in $\mathcal{E}_\lambda(\mathbb{C}(\lambda))$ modulo torsion. It is not difficult to show directly that $P_1 + T$,

P_2+T, P_1+P_2+T cannot lie in $2\mathcal{E}_\lambda(\mathbb{C}(\lambda))$ for any torsion element T, and thus P_1, P_2 generate $\mathcal{E}_\lambda(\mathbb{C}(\lambda))$ modulo torsion, as required.

To find a set of generators for $\mathrm{NS}(\mathcal{S},\mathbb{C})$ over $\mathbb{Z}$, we use ideas of Swinnerton-Dyer [16], to which article the reader is referred for full details (see also Bremner [2], [4]) for further applications of these methods). The group $\mathrm{NS}(\mathcal{S},\mathbb{C})$ is spanned over $\mathbb{Z}$ by

1. the locus of the point $\mathcal{O}_\lambda$,
2. the components of the singular fibres in the pencil $\mathcal{E}_\lambda$,
3. the loci of the generators P_1, P_2, T_4, T_2 of the group $\mathcal{E}_\lambda(\mathbb{C}(\lambda))$.

Now the locus of $\mathcal{O}_\lambda$ as λ varies is the conic C_{TWX}; the locus of P_1 is the conic C_{UVY}; and the locus of P_2 is the straight line

$$L_2: \quad \{V-X=U,\ V+X=-\sqrt{3}\,T,\ W-Y=T,\ W+Y=\sqrt{3}\,U\}.$$

From (22), the loci of T_4, T_2 are the conics C_{TVY} and C_{WXU} respectively. Together with the components of the singular fibres of $\mathcal{E}_\lambda$, from (16)–(20), the following divisors therefore generate $\mathrm{NS}(\mathcal{S},\mathbb{C})$ over $\mathbb{Z}$:

(23) $C_{TVY}, C_{TWX}, C_{TYV}, C_{TYW}, C_{UVY}, C_{UXV}, C_{UXW}, C_{WXU}$,
$C_{++-}, C_{+--}, C_{-++}, C_{--+}; L_2;$ the 4 lines at (20) (with symmetries).

The intersection matrix of these 29 divisors is straightforward to write down, and as expected, has rank 20. By repeatedly discarding a divisor that is a $\mathbb{Z}$-linear combination of remaining divisors, the following result is obtained.

Theorem. *The following* 20 *divisors generate* $\mathrm{NS}(\mathcal{S},\mathbb{C})$ *over* $\mathbb{Z}$: C_{TVX}, C_{UWX}, C_{TWX}, C_{UVX}, C_{TVY}, C_{VXT}, C_{WYT}, C_{WXT}, C_{VXU}, C_{UXV}, C_{TYV}, C_{TXV}, C_{+++}, L_2 (*which we denote by* $\Gamma_1,\ldots,\Gamma_{14}$, *respectively*), *and the* 6 *lines*:

$$(24)\quad \begin{array}{llllll}
\Gamma_{15}: & \{T+X=V, & T-X=\sqrt{3}\,W, & U+Y=W, & U-Y=\sqrt{3}\,V\}, \\
\Gamma_{16}: & \{T+X=-V, & T-X=\sqrt{3}\,W, & U+Y=-W, & U-Y=-\sqrt{3}\,V\}, \\
\Gamma_{17}: & \{T+X=W, & T-X=-\sqrt{3}\,V, & U+Y=V, & U-Y=\sqrt{3}\,W\}, \\
\Gamma_{18}: & \{T+X=\sqrt{3}\,W, & T-X=V, & U+Y=-\sqrt{3}\,V, & U-Y=W\}, \\
\Gamma_{19}: & \{T+X=-\sqrt{3}\,W, & T-X=-V, & U+Y=\sqrt{3}\,V, & U-Y=-W\}, \\
\Gamma_{20}: & \{T+X=\sqrt{3}\,V, & T-X=W, & U+Y=-\sqrt{3}\,W, & U-Y=V\}.
\end{array}$$

Corollary. *Denote by* $\mathrm{NS}(\mathcal{S},\mathbb{Q})$ *that subgroup of* $\mathrm{NS}(\mathcal{S},\mathbb{C})$ *which is defined over* $\mathbb{Q}$. *Then* $\mathrm{NS}(\mathcal{S},\mathbb{Q})$ *is generated over* $\mathbb{Z}$ *by the twelve divisors* $\Gamma_1,\ldots,\Gamma_{12}$.

Corollary. *Any rational curve on* $\mathcal{S}$ *has even degree.*

Remark. This corollary is mildly inconvenient for the following reason. We are interested in obtaining one-parameter solutions of the equations for $\mathcal{S}$, which geometrically represent irreducible curves of genus 0 lying on $\mathcal{S}$.

But an irreducible curve of genus 0 of even degree may not in fact actually be rationally parametrizable.

To any curve Γ defined over $\mathbb{Q}$ on $\mathcal{S}$ there thus correspond uniquely determined integers $m_1,\ldots,m_{12}$ such that $\Gamma \sim m_1\Gamma_1 + \ldots + m_{12}\Gamma_{12}$. The genus of Γ is a quadratic form in the m_i, given by

$$p_a(\Gamma) = \frac{1}{2}(\Gamma.\Gamma) + 1$$

where $(\Gamma.\Gamma)$ is the self-intersection number (see Shafarevich [12], p. 5). Following simple algebra, there results

$$\begin{aligned}(25)\quad & \frac{1}{2}\deg(\Gamma)^2 - 4(\Gamma.\Gamma)\\ &= (m_1+m_2-m_3-m_4-m_5-m_6+m_8+m_9+2m_{10})^2\\ &+ (m_1+m_2-m_3-m_4-m_5-m_6+m_8+m_9+2m_{11}-2m_{12})^2\\ &+ 2(m_1-m_4-m_5+m_6-m_8-m_9+m_{10}+m_{11}-m_{12})^2\\ &+ 2(m_1-m_4-m_5-m_6-m_7+m_8+m_9-m_{10}-m_{11}+m_{12})^2\\ &+ 4(m_1-m_3)^2+3(m_2-m_3)^2+(m_2-m_3-2m_7)^2\\ &+ 4(m_2-m_4+m_5)^2+4m_6^2+4(m_7-m_8+m_9)^2+2(m_{10}-m_{11}-m_{12})^2.\end{aligned}$$

This is now in a form suitable for machine computation. Given the degree and self-intersection number of Γ, it is possible to tabulate the finitely many sets of integers $m_1,\ldots,m_{12}$ that are solutions of (25). In addition, since we are only interested in irreducible curves Γ, further restrictions are imposed on the m_i by insisting that Γ have non-negative intersection number with every known curve lying on the surface. In this manner, it is computed first that the only irreducible rational curves on $\mathcal{S}$ of degree 2 are the known conics (which correspond to trivial squares), and second that there are no irreducible rational curves of degree 4. Of degree 6, there is the curve represented by the divisor $\Gamma_4+\Gamma_5+\Gamma_8+\Gamma_9-\Gamma_{12}$, together with 96 symmetries. This sextic is parametrized as follows:

$$\begin{aligned}(26)\quad T:U:V:W:X:Y = \quad & 2\lambda(7+4\lambda-\lambda^2)(1+2\lambda-3\lambda^2+2\lambda^3):\\ & (1-\lambda)(5+8\lambda+\lambda^2)(1+\lambda+5\lambda^2-3\lambda^3):\\ & -4(1+2\lambda-\lambda^2)(1-2\lambda-\lambda^2)^2:\\ & 3(1+\lambda^2)(1+8\lambda+2\lambda^2-8\lambda^3+\lambda^4):\\ & 2(1+4\lambda-7\lambda^2)(2+3\lambda+2\lambda^2-\lambda^3):\\ & (1+\lambda)(1-8\lambda+5\lambda^2)(3+5\lambda-\lambda^2+\lambda^3),\end{aligned}$$

and leads to the square with entries

$$m_{11} = (5+22\lambda+57\lambda^2-36\lambda^3-45\lambda^4+38\lambda^5-\lambda^6)^2,$$

$$
\begin{aligned}
m_{12} &= (1-2\lambda-\lambda^2)^2(7+20\lambda+2\lambda^2+4\lambda^3-5\lambda^4)^2,\\
m_{13} &= (1-32\lambda-37\lambda^2+48\lambda^3+19\lambda^4+16\lambda^5-7\lambda^6)^2,\\
m_{31} &= (7+16\lambda-19\lambda^2+48\lambda^3+37\lambda^4-32\lambda^5-\lambda^6)^2,\\
m_{32} &= (1+38\lambda+45\lambda^2-36\lambda^3-57\lambda^4+22\lambda^5-5\lambda^6)^2,\\
m_{33} &= (1-2\lambda-\lambda^2)^2(5+4\lambda-2\lambda^2+20\lambda^3-7\lambda^4)^2
\end{aligned}
$$

and middle row elements

$$
\begin{aligned}
m_{21} &= (1-2\lambda-\lambda^2)^2\\
&\quad\times(1-200\lambda-436\lambda^2+232\lambda^3+94\lambda^4-88\lambda^5+860\lambda^6-520\lambda^7+73\lambda^8),\\
m_{22} &= (5-4\lambda+\lambda^2)(1+4\lambda+5\lambda^2)(1+18\lambda^2+\lambda^4)(5-6\lambda^2+5\lambda^4),\\
m_{23} &= (7+38\lambda-9\lambda^2-36\lambda^3+57\lambda^4+22\lambda^5-23\lambda^6)\\
&\quad\times(7+14\lambda+15\lambda^2+108\lambda^3-87\lambda^4-2\lambda^5+\lambda^6).
\end{aligned}
$$

A search for values of λ making one of these latter three entries square disclosed only $\lambda=\pm1$, giving trivial cases.

There are no irreducible rational curves of degree 8 on $\mathcal{S}$, and up to symmetry just one curve of degree 10, given by the divisor $2\Gamma_1+2\Gamma_3+\Gamma_4+\Gamma_6-2\Gamma_7-\Gamma_8+2\Gamma_9+\Gamma_{11}-\Gamma_{12}$ with parametrization

(27) $T:U:V:W:X:Y=$

$$
\begin{aligned}
&2(-1+\lambda)(-2-6\lambda+\lambda^2)(-2+10\lambda-11\lambda^2+4\lambda^3)(100-216\lambda+128\lambda^2-36\lambda^3+\lambda^4):\\
&(22-6\lambda+\lambda^2)(-8+18\lambda-14\lambda^2+3\lambda^3)(72-220\lambda+264\lambda^2-152\lambda^3+30\lambda^4+5\lambda^5):\\
&(14-30\lambda+17\lambda^2)(-4+10\lambda-4\lambda^2+\lambda^3)(112-188\lambda+104\lambda^2-16\lambda^3-8\lambda^4+\lambda^5):\\
&2(10-18\lambda+7\lambda^2)(-6+4\lambda-3\lambda^2+\lambda^3)(92-264\lambda+304\lambda^2-156\lambda^3+23\lambda^4):\\
&3(2-2\lambda+\lambda^2)(20-40\lambda+24\lambda^2-4\lambda^3+\lambda^4)(68-264\lambda+280\lambda^2-84\lambda^3+5\lambda^4):\\
&-8(-2+\lambda^2)(4-6\lambda+\lambda^2)(-152+480\lambda-636\lambda^2+432\lambda^3-150\lambda^4+24\lambda^5+\lambda^6).
\end{aligned}
$$

There are no irreducible rational curves of degree 12, and, up to symmetry, 6 of degree 14 corresponding to the divisors:

$$
\begin{gathered}
\Gamma_3-\Gamma_4+2\Gamma_5+\Gamma_6-\Gamma_7+2\Gamma_8+2\Gamma_{11}+\Gamma_{12},\\
\Gamma_3-\Gamma_4-\Gamma_5+\Gamma_7+\Gamma_8+3\Gamma_{10}+3\Gamma_{12},\\
\Gamma_1+3\Gamma_2+\Gamma_3+\Gamma_4-2\Gamma_5+2\Gamma_6+\Gamma_7-\Gamma_8+\Gamma_9,\\
\Gamma_2-\Gamma_5+3\Gamma_6+3\Gamma_9+\Gamma_{11},\\
-2\Gamma_7+2\Gamma_9+\Gamma_{10}+3\Gamma_{11}+3\Gamma_{12},\\
\Gamma_5+\Gamma_7-\Gamma_9+3\Gamma_{11}+3\Gamma_{12}.
\end{gathered}
$$

It becomes impracticable to compute the zeros of the form (25) for degrees greater than 14; and in any event, deciding whether the divisors found in this way represent irreducible curves becomes increasingly difficult. The manner in which we have resolved this for the divisors above is as follows.

As in Swinnerton-Dyer [16] it is most useful to introduce non-linear automorphisms of the surface.

Let $\mathcal{E}$ denote a pencil of curves on $\mathcal{S}$ of genus 1, the general member of which is irreducible; and let C_1, C_2 be two curves on $\mathcal{S}$ each having precisely one point of intersection with any member of $\mathcal{E}$. For P a generic point of $\mathcal{S}$, let $\mathcal{E}_P$ be that member of $\mathcal{E}$ which passes through P, and let P_1, P_2 be the points of $\mathcal{E}_P$ in which C_1, C_2 intersect $\mathcal{E}_P$. Then

$$P \mapsto P_1 + P_2 - P,$$

where the addition is that of the group law on $\mathcal{E}_P$, defines a birational map of $\mathcal{S}$ to itself. Such a map is necessarily biregular (Shafarevich [12], Chapter VII, Corollary to Theorem 1), and hence gives an automorphism of $\mathcal{S}$. Moreover, it is clear from the definition that this automorphism is actually an involution. The involution certainly interchanges the curves C_1, C_2, and preserves each non-singular fibre of $\mathcal{E}$, so by biregularity, actually preserves every fibre of $\mathcal{E}$. So if the fibre is singular, then the involution permutes the components of the decomposition, in particular interchanging the component containing P_1 with the component containing P_2. Further information about the action on the Néron–Severi group is obtained from the fact that the involution has fixed points on each non-singular fibre (for there are just four points P satisfying $2P = P_1 + P_2$), and hence, by specialization, at least one fixed point on every fibre. In general, the components of $\mathcal{E}$ together with C_1 and C_2 span only a subgroup $\mathcal{G}$ of the Néron–Severi group, so there is not yet sufficient information to determine fully the action of the involution. However, Bremner [2] shows that the involution actually must reverse every element of the orthogonal complement (with respect to intersection) of $\mathcal{G}$ in the Néron–Severi group; and this in general now allows computation of the action of the involution. Indeed, the parametrizations of the curves of degrees 6 and 10 above were computed in exactly this way, as the image of a conic under an involution chosen with appropriate C_1, C_2.

As an example, we observe that up to symmetry there is on $\mathcal{S}$ just one other elliptic pencil $\mathcal{F}_\lambda$ besides $\mathcal{E}_\lambda$ of degree 4 (put $\deg(\Gamma) = 4$ and $(\Gamma.\Gamma) = 0$ in (25)), with the divisor $\Gamma_1 + \Gamma_3$. It is the residual intersection of $\mathcal{S}$ with the hyperplane

$$T - U + V + W = \lambda X,$$

after removing the conics C_{UVX}, C_{UWX}. $\mathcal{F}_\lambda$ is singular at $\lambda = \pm 2\sqrt{2}, \pm 1/\sqrt{2}$ (nodal quartics), together with the following decompositions:

$$\mathcal{F}_0: \quad C_{UVY} + C_{UWY}, \qquad \mathcal{F}_\infty: \quad C_{TVX} + C_{TWX},$$

$$\begin{aligned}\mathcal{F}_{1+\sqrt{3}}:\quad &\{U-X=-\sqrt{3}\,W,\ U+X=V,\quad T-Y=\sqrt{3}\,V,\ T+Y=W\}\\ &+\{U-X=-\sqrt{3}\,V,\ U+X=W,\quad T-Y=\sqrt{3}\,W,\ T+Y=V\}\\ &+\{T-X=V,\quad T+X=\sqrt{3}\,W,\ U-Y=-W,\quad U+Y=-\sqrt{3}\,V\}\\ &+\{T-X=-W,\quad T+X=\sqrt{3}\,V,\ U-Y=-V,\quad U+Y=-\sqrt{3}\,W\}\end{aligned}$$

with conjugate decomposition at $\lambda = 1-\sqrt{3}$, and symmetric decompositions at $\lambda = -1 \pm \sqrt{3}$. Choosing C_1, C_2 as the conics C_{WXT}, C_{WYT} respectively, results in the following involution, where we represent the action by means of a 12×12 matrix on $\Gamma_1, \ldots, \Gamma_{12}$ as basis:

$$\phi:\ \begin{pmatrix} 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&-1&0&-1&1&0&1&1&0&1&0&1\\ 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&-1&0&1&1&0&1&0&1\\ 0&0&0&-1&0&0&1&1&0&1&0&1\\ 0&-1&0&-1&0&1&0&1&1&1&0&1\\ 0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&1&-1&0&0&0\\ 0&-1&0&0&1&0&-1&0&1&1&0&0\\ 1&0&1&0&0&0&1&1&0&0&-1&0\\ 0&1&0&0&-1&0&1&0&-1&0&0&1 \end{pmatrix}$$

(so $\phi(\Gamma_1) = \Gamma_3$, etc.)

The image of the conic C_{TWY} (with divisor $\Gamma_2 + \Gamma_4 - \Gamma_5$) is the sextic curve with divisor $-\Gamma_2 + \Gamma_7 + \Gamma_8 + \Gamma_{10} + \Gamma_{12}$, whence the parametrization (26). Similarly, the sextic $\Gamma_1 + \Gamma_4 + \Gamma_6 + \Gamma_8 - \Gamma_{11}$ maps under the involution to the divisor $-\Gamma_1 - \Gamma_2 - \Gamma_4 - \Gamma_5 + \Gamma_6 + \Gamma_7 + \Gamma_8 + \Gamma_9 + 2\Gamma_{10} + \Gamma_{11} + 2\Gamma_{12}$ of degree 10, producing the parametrization at (27). The involution may also be useful in spotting reducibility of a divisor. For example, up to symmetry, just one divisor arises from (25) of degree 12 and self-intersection -2, namely $-\Gamma_2 + \Gamma_3 - \Gamma_4 + \Gamma_7 + \Gamma_8 + \Gamma_9 + 2\Gamma_{10} + 2\Gamma_{12}$. However, under ϕ, this divisor maps to the sum of Γ_{10} and the line pair $\{U - X = -V,\ U + X = \mp\sqrt{3}\,W,\ T - Y = -W,\ T + Y = \pm\sqrt{3}\,V\}$, and accordingly is reducible.

A theorem whose proof was sketched by E. Looijenga (see Sterk [15]) proves that all curves of genus 0 on $\mathcal{S}$ are obtainable by possibly repeated application of a finite set of automorphisms of the surface to one of a finite number of curves of genus 0 on the surface. Swinnerton-Dyer [16] proved such a theorem for the quartic $K3$ surface $A^4 + B^4 = C^4 + D^4$ with explicit determination of the automorphisms (two in number, together with the symmetries) and set of base curves comprising the straight line $A = C, B = D$. For an explicit such calculation on another quartic surface (contained in the four-fold $x^5 + y^5 + z^5 = u^5 + v^5 + w^5$), see Bremner [4]. In similar manner it should be possible, in principle at least, to construct the relevant sets of automorphisms and base curves for the surface $\mathcal{S}$. In practice however, the computation is sufficiently daunting that it has not been pursued. (We remark that at least two further automorphisms of $\mathcal{S}$ seem to be needed in order to generate the six curves of degree 14.)

4. We turn finally to a brief investigation over the ground field $\mathbb{Q}(\sqrt{3}\,)$. The line at (14) corresponds to the square

$$\begin{bmatrix} ((1+\sqrt{3}\,)\alpha+2\beta)^2 & (2\alpha+(1-\sqrt{3}\,)\beta)^2 & ((1-\sqrt{3}\,)\alpha-(1+\sqrt{3}\,)\beta)^2 \\ * & * & * \\ ((1+\sqrt{3}\,)\alpha-(1-\sqrt{3}\,)\beta)^2 & (2\alpha+(1+\sqrt{3}\,)\beta)^2 & ((1-\sqrt{3}\,)\alpha+2\beta)^2 \end{bmatrix} \tag{28}$$

where the middle row elements are

$$\begin{aligned} m_{21} &= 4(1-\sqrt{3}\,)\alpha^2 + 4(1-\sqrt{3}\,)\alpha\beta + (1+\sqrt{3}\,)^2\beta^2, \\ m_{22} &= 4\alpha^2 + 4\alpha\beta + 4\beta^2, \\ m_{23} &= 4(1+\sqrt{3}\,)\alpha^2 + 4(1+\sqrt{3}\,)\alpha\beta + (1-\sqrt{3}\,)^2\beta^2. \end{aligned} \tag{29}$$

To achieve *two* of $\{m_{21}, m_{22}, m_{23}\}$ being perfect squares in $\mathbb{Q}(\sqrt{3}\,)$ is equivalent to finding points on an elliptic curve over $\mathbb{Q}(\sqrt{3}\,)$. For example, $m_{21} = \square, m_{23} = \square$ demands

$$\begin{cases} 4(1-\sqrt{3}\,)\alpha^2 + 4(1-\sqrt{3}\,)\alpha\beta + (1+\sqrt{3}\,)^2\beta^2 = \square, \\ 4(1+\sqrt{3}\,)\alpha^2 + 4(1+\sqrt{3}\,)\alpha\beta + (1-\sqrt{3}\,)^2\beta^2 = \square \end{cases}$$

which is the equation of an elliptic curve having cubic model

$$Y^2 = X(X^2 + 8X + 4).$$

This curve is of rank 1 over $\mathbb{Q}(\sqrt{3}\,)$ and has generator of infinite order equal to $P = (-1, \sqrt{3}\,)$.

The multiples of P lead to an infinite sequence of squares such as (2). Points Q and Q' that differ only by a torsion element lead to symmetries of the same magic square, and so magic squares that arise in this way come from the sequence mP, $m \in \mathbb{Z}$. For example, P corresponds to $(\alpha, \beta) = (2, -1)$, giving a trivial square; $2P$ corresponds to $(\alpha, \beta) = (4, 9)$, leading to the square (2); and $3P$ corresponds to $(\alpha, \beta) = (2926, -3041)$, giving the square

$$\begin{bmatrix} (3156 - 2926\sqrt{3}\,)^2 & (2811 + 3041\sqrt{3}\,)^2 & (5967 + 115\sqrt{3}\,)^2 \\ (4749 + 2089\sqrt{3}\,)^2 & 2^2 \cdot 3 \cdot 37 \cdot 43 \cdot 1867 & (4749 - 2089\sqrt{3}\,)^2 \\ (5967 - 115\sqrt{3}\,)^2 & (2811 - 3041\sqrt{3}\,)^2 & (3156 + 2926\sqrt{3}\,)^2 \end{bmatrix}.$$

The other possibility, that $m_{21} = \square, m_{22} = \square$, demands

$$\begin{cases} 4(1-\sqrt{3}\,)\alpha^2 + 4(1-\sqrt{3}\,)\alpha\beta + (1+\sqrt{3}\,)^2\beta^2 = \square, \\ \hfill \alpha^2 + \alpha\beta + \beta^2 = \square. \end{cases}$$

This is the equation of an elliptic curve over $\mathbb{Q}(\sqrt{3}\,)$, with cubic model

$$Y^2 = X(X^2 + 2X - 2). \tag{30}$$

This curve is of rank 2 over $\mathbb{Q}(\sqrt{3}\,)$ and has generators of infinite order equal to $P_1 = (1,1)$, $P_2 = (-1, \sqrt{3}\,)$. A two-dimensional family of magic squares arises from the pullbacks of combinations of the two generators. For example, P_1 gives rise to

$$\begin{bmatrix} (23-7\sqrt{3}\,)^2 & (1-4\sqrt{3}\,)^2 & (22-3\sqrt{3}\,)^2 \\ (2+9\sqrt{3}\,)^2 & (7-11\sqrt{3}\,)^2 & 577-344\sqrt{3} \\ (11-8\sqrt{3}\,)^2 & 5^2(2-3\sqrt{3}\,)^2 & (1+7\sqrt{3}\,)^2 \end{bmatrix},$$

P_2 to

$$\begin{bmatrix} (5+3\sqrt{3}\,)^2 & (5-4\sqrt{3}\,)^2 & (10-\sqrt{3}\,)^2 \\ (10-3\sqrt{3}\,)^2 & (1-5\sqrt{3}\,)^2 & 5(5+8\sqrt{3}\,) \\ 7^2 & (2+5\sqrt{3}\,)^2 & 5^2(1-\sqrt{3}\,)^2 \end{bmatrix},$$

$P_1 + P_2$ to

$$\begin{bmatrix} (95-17\sqrt{3}\,)^2 & (205-68\sqrt{3}\,)^2 & (110-51\sqrt{3}\,)^2 \\ 5^2(26-17\sqrt{3}\,)^2 & (83-85\sqrt{3}\,)^2 & 18553-6120\sqrt{3} \\ 5^2(17-20\sqrt{3}\,)^2 & (34-5\sqrt{3}\,)^2 & 7^2(17-15\sqrt{3}\,)^2 \end{bmatrix},$$

and $2P_2$ to

$$\begin{bmatrix} (187-243\sqrt{3}\,)^2 & (166+203\sqrt{3}\,)^2 & (353-40\sqrt{3}\,)^2 \\ (227+100\sqrt{3}\,)^2 & (37-233\sqrt{3}\,)^2 & 317(779-252\sqrt{3}\,) \\ (446-7\sqrt{3}\,)^2 & (283-180\sqrt{3}\,)^2 & (163+173\sqrt{3}\,)^2 \end{bmatrix}.$$

None of the squares we computed had the ninth element m_{23} a perfect square in $\mathbb{Q}(\sqrt{3}\,)$.

5. As a final remark, we observe that non-trivial 4×4 magic squares of squares are not difficult to construct. One such example is the following:

$$\begin{bmatrix} 37^2 & 23^2 & 21^2 & 22^2 \\ 1^2 & 18^2 & 47^2 & 17^2 \\ 38^2 & 11^2 & 13^2 & 33^2 \\ 3^2 & 43^2 & 2^2 & 31^2 \end{bmatrix}.$$

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Department of Mathematics
Arizona State University
Tempe, AZ 85287-1804, U.S.A.
E-mail: bremner@asu.edu

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